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Π_2^1-LOGIC, PART 1: DILATORS

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Foreword

The expression 'Π_2^1-logic' must be defined somewhere: so let's add this term to the dictionary:

Π_2^1-*logic:* Part of logic concerned with the development of Π_2^1 concepts such as dilators, β-proofs, by means of certain kinds of functors commuting to direct limits and to pull-backs. The theory is developed in analogy with ω-logic (i.e., Π_1^1-logic), and the name comes from the fact that the concepts of Π_2^1-logic are Π_2^1-universal.

Let us add a word of warning to the reader that the title will not at all be justified in this part I, mainly because all logical questions (recursivity, β-completeness, Π_2^1-universality...) are relegated to part II.

Although the main lines of Π_2^1-logic have remained the same since its invention in 75–76, there are many essential technical differences, which have been reflected by successive change of concepts. (76–77: gardens, 78: first concept of dilator, 78–80: ladders, now: dilators).

We have tried to give an unified approach: this means that we have looked for:

(1) A main concept (*dilators*) easy to apprehend from the most successful of viewpoints, i.e., from set theory.

(2) Alternative concepts (*dendroids*): we have in mind the idea that the situation is fruitful, because:

– the concepts (dendroids and dilators) are very different: so they correspond to diverging needs

– but they are equivalent (and not only vaguely related); so, everytime one proves a result for dilators, this induces a result on strongly homogeneous dendroids, and conversely: the intrinsic interest of each approach is enhanced by the equivalence with the other approach.

I have tried to give detailed and complete proofs. But there is one noticeable exception: Section 7. The purpose of this section is to rework the contents of Sections 2–5 in terms of the new notions introduced here (multi-dendroids, for instance), and it was not possible to give systematic proofs. We hope that the reader will forgive us; the repetition of all proofs given for dilators in this context would in any case not have brought much.

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Another question is the use of categories and of category-theoretic language. With respect to this problem, I have always adopted a moderate position, between those who prefer to expell categories from mathematics, and those who think only in terms of diagrams. My position was not to drown the reader in an ocean of abstract concepts (for instance, the dilators, as a category, are treated only in Section 4, in order to leave some time for the reader to become familiarized with dilators), and to use direct arguments (and not diagrams) when possible: this is the reason of the extreme importance given to the normal form theorem of Section 2 (the normal form theorem is a non-category-theoretic treatment of commutation to & and $\varinjlim$). Non category theoretic issues, such as multi-dendroids, are also developed. However, we think that, without a minimum of categories, this work would be impossible to understand. But we have systematically refrained from making 'abstract generalizations'. (For instance, sums and products of dilators are functors from ad hoc categories of sequences of dilators (or regular bilators): in that case, we have not introduced these categories). One more word for specialists of categories: our terminology is slightly incorrect: a true category theorist would have said 'preserves' at every place I have said 'commutes', because the term commutation means that in the corresponding categories, the kind of limit considered always exists: this is practically never true for direct limits, and sometimes false for pull-backs.

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0. Introduction

0.1. A theory of ordinal denotation

We shall use the term '*ordinal denotation*' (in opposition with 'ordinal notation', to mean the act of representing an ordinal by means of other ordinals (in general smaller). An approach to dilators is that they are the *theory of ordinal denotation.*

0.1.1. Systems of ordinal denotations

One must take examples from usual practice:

(a) The Cantor Normal Form of base, say 10, induces a system of ordinal denotation: any ordinal can be written

$$z = 10^{x_0} \cdot a_0 + \cdots + 10^{x_{n-1}} \cdot a_{n-1}$$

where $a_0, \ldots, a_{n-1}$ are non-zero integers <10, and $x_0, \ldots, x_{n-1}$ are ordinals, with $x_0 > x_1 > \cdots > x_{n-1}$.

(b) But also the representation of ordinals $<x^2$: any ordinal $<x^2$ can be written:

$$z = x \cdot x_0 + x_1, \quad \text{with } x_0, x_1 < x.$$

We must now decide, in these examples, which are the accidental and the inherent features of ordinal denotation. For instance, it is clear that the spacial two-dimensional explicit writing of the denotation is contingent; on the other hand the list of the ordinals parameters occuring in it is certainly not. Concerning the ordinal parameters, one can notice an important difference between (a) and (b): the denotation (a) is universal, while denotation (b) depends on an x such that $z < x^2$. One decides that the situation of (b) is the usual one, and it is only by chance that the representation of ordinals $<10^x$ by means of ordinals $\leq x$ does not depend on x.

We shall represent the situation as follows: a system of ordinal denotation permits one to represent any ordinal $<F(x)$ by means of a sequence $(C; x_0, \ldots, x_{n-1}; x)$, where:

(i) x is given in advance (in (a), x is not present);

(ii) $x_0 < \cdots < x_{n-1} < x$ (this corresponds to the list of ordinal parameters $\neq x$, in increasing order);

(iii) C is the 'configuration', i.e., what remains of the denotation, when abstracted from its ordinal parameters.

It will be possible to make a theory, provided one adds some very natural properties of denotations:

(1) The representation $z = (C; x_0, \ldots, x_{n-1}, x)$ exists and is unique, for all ordinals $z < F(x)$.

(2) If $(C; x_0, \ldots, x_{n-1}; x)$ is a denotation, if $x'_0 < \cdots < x'_{n-1} < x'$ s another sequence, then $(C; x'_0, \ldots, x'_{n-1}; x')$ is a denotation. (This means that, in one of our familiar systems, like (a) and (b), it is possible to change the ordinal parameters, provided one respects their mutual order: the result is still a denotation.)

(3) If $(C; x_0, \ldots, x_{n-1}; x) < (C'; y_0, \ldots, y_{m-1}; x)$, and if $x'_0 < \cdots < x'_{n-1} < x'$, $y'_0 < \cdots < y'_{m-1} < x'$ are such that $x_i < y_j$ iff $x'_i < y'_j$ for $i = 0, \ldots, n-1$, $j = 0, \ldots, m-1$, then

$$(C; x'_0, \ldots, x'_{n-1}; x') < (C'; y'_0, \ldots, y_{m-1}; x')$$

(In familiar systems, like (a) and (b), this means that the mutual order between two denotations *with the same* x depends only on the mutual order between their parameters).

0.1.2. Dilators

A system of ordinal denotations induces a function from ON to ON: $F(x)$ is the set of all ordinals admitting a representation $z=(C;x_0,\ldots,x_{n-1};x)$ in the system. But, indeed, F can be considered as a *functor* from ON to ON (see Section 1.1 for the definition of the category ON): if $f\in I(x,y)$, then it is possible to define a strictly increasing function $F(f)\in I(F(x),F(y))$ by: write $z=(C;x_0,\ldots,x_{n-1};x)$; then $(C;f(x_0),\ldots,f(x_{n-1});y)$ is still a denotation, by property (2) of systems of denotation: let $F(f)(z)=$ ordinal denoted by $(C;f(x_0),\ldots,f(x_{n-1});y)$. In short

$$F(f)(C;x_0,\ldots,x_{n-1};x)=(C;f(x_0),\ldots,f(x_{n-1});y).$$

Property (3) of denotations ensures that $F(f)$ is strictly increasing: if $(C;x_0,\ldots,x_{n-1};x)<(C';y_0,\ldots,y_{n-1};x)$, then, by (3),

$$(C;f(x_0),\ldots,f(x_{n-1});y)<(C';f(y_0),\ldots,f(y_{n-1});y).$$

We shall see that the associated functor *characterizes* the system of denotation, and this will justify replacing the study of systems of denotation by the study of their associated functors.

The functor F has two properties:

(1) If $z<F(x)$, then one can find an integer n, $z_0<F(n)$ and $f\in I(n,x)$, such that $z=F(f)(z_0)$. This property is exactly *commutation to direct limits.* (See Section 2.1, especially Corollary 2.1.8.) One can establish it as follows: write $z=(C;x_0,\ldots,x_{k-1};x)$, then let $n=k$, $z_0=(C;0,\ldots,k-1;k)$, and define $f\in I(k,x)$ by $f(0)=x_0,\ldots,f(k-1)=x_{k-1}$. Then, by definition of $F(f)$, $F(f)(z_0)=z$.

(2) Assume that we have three morphisms f,g,h, with target x, and that $\mathrm{rg}(f)\cap\mathrm{rg}(g)=\mathrm{rg}(h)$; then $\mathrm{rg}(F(f))\cap\mathrm{rg}(F(g))=\mathrm{rg}(F(h))$. This property is exactly *commutation to pull-backs* (see Section 2.2). We prove it as follows: $(C;x_0,\ldots,x_{n-1};x)$ belongs to $\mathrm{rg}(F(f))$ iff all parameters $x_0,\ldots,x_{n-1}$ belong to $\mathrm{rg}(f)$. This comes from, when $f\in I(y,x)$

$$F(f)(C;y_0,\ldots,y_{n-1};y)=(C;f(y_0),\ldots,f(y_{n-1});x)$$

From this, $(C;x_0,\ldots,x_{n-1};x)\in\mathrm{rg}(F(f))\cap\mathrm{rg}(F(g))$ iff $x_0,\ldots,x_{n-1}$ are in both $\mathrm{rg}(f)$ and $\mathrm{rg}(g)$, i.e., are in $\mathrm{rg}(h)$: this is equivalent to $(C;x_0,\ldots,x_{n-1};x)\in\mathrm{rg}(F(h))$.

A functor from ON to ON commuting to direct limits and to pull-backs is called a *dilator*; so, a system of ordinal denotation induces a dilator. We show now that the converse is true: all dilators can be obtained in that way, and the dilator determines the system of denotation completely.

0.1.3. The normal form theorem

One will find a proof in Section 2.3: assume that F is a dilator, then it is possible to represent any $z < F(x)$ as $z = (z_0; x_0, \ldots, x_{n-1}; x)$, with

$$f \in I(n, x), \qquad z_0 < F(n), \qquad z = F(f)(z_0)$$
$$\text{and} \quad f(0) = x_0, \ldots, f(n-1) = x_{n-1};$$

the denotation is made unique by requiring 'n is the minimal such'.

Observe that this defines a system of denotation, in the precise sense of Section 0.1.1: the configurations are all pairs $(z_0; n)$ such that if $f \in I(m, n)$ and $z_0 \in \mathrm{rg}(F(f))$, then $m = n$. (The set of configurations plays an essential role, and is called the *range of F*.) Configuration $(z_0; n)$ permits us to construct a denotation, when one adds an arbitrary sequence $x_0 < \cdots < x_{n-1} < x$. Properties (1) and (2) of systems of ordinal denotation are satisfied. Property (3) holds too, as proved in Proposition 2.3.17.

So, a dilator induces, by means of the normal form theorem, a system of ordinal denotations. It is clear that the mappings we have constructed from systems of ordinal denotation to dilators and *vice versa* are inverses of one another (provided one identifies isomorphic systems: in the normal form theorem, configuration C (which needs the parameters $x_0, \ldots, x_{n-1}, x$ to make a denotation) is denoted by $(z_0; n)$: n is the number of parameters $\neq x$, and $z_0 = (C; 0, \ldots, n-1; n)$ is the 'smallest' example of a notation with configuration C).

0.1.4. Fine structure of the range

We work now on denotation systems associated with dilators; the order between $(z_0; x_0, \ldots, x_{n-1}; x)$ and $(z_1; y_0, \ldots, y_{m-1}; x)$ depends only on the relative order of the x_i's and y_j's; another question is 'how?'. In a usual system of denotations, a *mode d'emploi* is supplied with the system, for instance, if one wants to compare $xa + b$ and $xa' + b'$, then first look at the first coefficients $a, a', \ldots$.

(i) comparison of $(z_0; x_0, \ldots, x_{n-1}; x)$ with $(z_0; y_0, \ldots, y_{n-1}; x)$: the configuration is the same. To each configuration $(z_0; n) \in \mathrm{rg}(F)$, one associates a permutation $\sigma_{z_0, n}$ of n (see Section 3.2): the permutation lists the coefficients $x_0, \ldots, x_{n-1}$, in a different order corresponding to their importance in the denotation (in $xa + b$, the coefficient a is 'more important' than b). The idea is to form the points $s_i = (z_0; 0, 2, \ldots, 2i-2, 2i+1, 2i+2, \ldots, 2n-2; 2n)$, for $i = 0, \ldots, n-1$; these points s_i (which correspond to the idea of slight increase of the ith coefficient) are pairwise distinct, so one defines σ by:

$$\sigma(i) < \sigma(j) \quad \text{iff } s_i > s_j.$$

The mode d'emploi of the permutation is simple: assume that $x_{\sigma(0)} = y_{\sigma(0)}, \ldots, x_{\sigma(k-1)} = y_{\sigma(k-1)}, x_{\sigma(k)} < y_{\sigma(k)}$, then

$$(z_0; x_0, \ldots, x_{n-1}; x) < (z_0; y_0, \ldots, y_{n-1}; x).$$

(ii) comparison of $(z_0; x_0, \ldots, x_{n-1}; x)$ with $(z_1; y_0, \ldots, y_{m-1}; x)$, when $(z_0; n) \neq (z_1, m)$. The comparison is made possible by means of data

$\S(z_0, n; z_1, m) = (k, \pm)$, the first component being an integer $k \leq n, m$, and the second is the symbol $+$ or the symbol $-$; if $\S(z_0, n; z_1, m) = (k, +)$, then $\S(z_1, m; z_0, n) = (k, -)$, and conversely (see Section 6.4).

The mode d'emploi of $\S$ is simple too: let $\sigma = \sigma_{z_0, n}$, $\tau = \sigma_{z_1, m}$: then, if $\S(z_0, n; z_1, m) = (k, +)$ and $x_{\sigma(0)} = y_{\tau(0)}, \ldots, x_{\sigma(i-1)} = y_{\tau(i-1)}$, then $(z_0; x_0, \ldots, x_{n-1}; x) < (z_1; y_0, \ldots, y_{m-1}; x)$, in the two following situations:

(a) $i < k$ and $x_{\sigma(i)} < y_{\tau(i)}$;
(b) $i \geq k$.

0.1.5. Morphisms of dilators

Dilators make a category, when one takes as morphisms natural transformations. In terms of denotations, a natural transformation acts only on the configurations: if $T \in I(F, G)$, then

$$T(x)((z_0; x_0, \ldots, x_{n-1}; x)_F) = (T(n)(z_0); x_0, \ldots, x_{n-1}; x)_G.$$

In other terms, the natural transformation T is completely determined by the function t from $\mathrm{rg}(F)$ to $\mathrm{rg}(G)$ defined by:

$$t((z_0; n) = (T(n)(z_0); n).$$

The function t enjoys, when $C \in \mathrm{rg}(F) = \sigma_C^F = \sigma_{t(C)}^G$, and, similarly, $\S(C, C')^F = \S(t(C), t(C'))^G$. Conversely, given such a function t, there is an unique natural transformation T such that

$$t((z_0; n)) = (T(n)(z_0); n).$$

In practice, we shall not work with the function t, but with its range, denoted $\mathrm{rg}(T)$. Characterizations of direct limits and pull-backs in the category of dilators can be found in Section 4, and they are in perfect analogy with the characterizations of Section 1.

In Section 1, one of the main (and trivial) results is that any ordinal is a direct limit of integers (Theorem 1.3.9); in the category of dilators, the role of integers is played by *strongly finite* dilators, i.e., dilators with a finite range, and a similar result holds (Theorem 4.3.10). Strongly finite dilators exactly those dilators which are polynomial. So, in some sense, polynomials are 'dense' in dilators.

0.1.6. Flowers

When analysing ordinal denotation, recall that, in case (a) the ordinal x is not necessary; we call a dilator a *flower* when this holds, i.e., when $(z_0; x_0, \ldots, x_{n-1}; x)$ is independant of x. An alternative definition is that F preserves the inclusion maps E_{xy}:

$$F(E_{xy}) = E_{F(x)F(y)}.$$

Flowers are essential for the study of dilators: one of their main properties is that a flower induces a normal function from ON to ON: in this concept, topological continuity and categorical continuity are reconciliated. For more details concerning flowers, see Section 2.4.

0.2. A dynamic theory of ordinals

0.2.1. Dilators and finitism

The first obvious feature of dilators is their 'finitistic' character: if one wants to compute $F(x)$, then one needs only:
- the restriction of F to the category $\mathrm{ON}{<}\omega$ of integers;
- an effective description of x (eventually given by an oracle).

In particular if one considers *weakly finite* dilators (i.e., dilators such that $F(n)$ is finite for all n), we have obtained a perfectly finitistic tool, not less finitistic than, say, recursive ordinals.

The originality of the situation is the possibility of 'blowing up' or 'dilating' F up to an arbitrary size x, but always in a finitary way: we shall therefore be able to perform operations on proper classes, and at the same time being strictly finitistic! In that sense we are using ordinals in a *dynamic* way.

0.2.2. Dilators and ordinal classes

If one can compute $F(x)$ when x is an ordinal, then the same process gives $F(\mathrm{ON})$, where ON is the full ordinal line: $F(\mathrm{ON})$ is simply a direct limit taken along the proper class ON! In general $F(\mathrm{ON})$ (which is a well-order) is not a set, for instance, if $F = \mathrm{Id}+\mathrm{Id}$ ($\mathrm{Id}=$ identity), then $F(\mathrm{ON})$ has the order type $\mathrm{ON}+\mathrm{ON}$, etc... So $F(\mathrm{ON})$ is an ordinal class, and in fact F can be considered as a way of *defining the class $F(\mathrm{ON})$, as a function of* ON. If one prefers, the class $\mathrm{ON}+\mathrm{ON}$ has no meaning as a collection of points, but is better described by the dilator $\mathrm{Id}+\mathrm{Id}$, which, when a 'size' for ON is given, yields a particular well-ordered class $\mathrm{ON}+\mathrm{ON}$.

This approach is particularily interesting because, in many situations, one does not need the actual 'size' of ON, but simply that ON is greater than any ordinal occurring in the construction we are making: so it will suffice to work with $\mathrm{ON}=$ first ordinal greater than all those already used at the stage we are; if it happens that greater ordinals appear at a later stage, then simply dilate ON into ON'... Combining this with Section 0.2.1 above, it seems possible to keep a finitary control on classes $F(\mathrm{ON})$.

0.2.3. Induction on dilators

As explained in Section 0.2.2 , we try to work on $F(\mathrm{ON})$, without 'freezing' ON: the size must remain flexible. We express this by the *principle of induction on dilators*, which is the main goal of Section 3, and the heart of the whole paper.

The idea is to define a predecessor relation among dilators, which is well founded (more: the predecessors of a given dilator are linearily ordered) and has order type $F(\mathrm{ON})$; of course the class of predecessors of F will depend on which ordinals we recognize to be sets, but, since a dilator must map sets into sets, there is no contradiction here.

First one observes that a dilator can be written uniquely as a sum of non-zero and undecomposable dilators (*perfect* dilators). This induces a classification of dilators in four kinds, according to the length LH(F) of the family of perfect dilators in the decomposition of F.

kind 0: when LH(F) = 0: this corresponds to the null dilator $\underset{\sim}{0}$;
kind ω: when LH(F) is limit;
kind 1: when LH(F) is successor, and $F = F' + \underset{\sim}{1}$, for some F';
kind Ω: when LH(F) is successor, and $F = F' + F''$, F'' perfect $\neq \underset{\sim}{1}$.

If one considers F(ON) with the 'naive' meaning of ON, then kind 0 corresponds to $F(\mathrm{ON}) = 0$, kind ω to the case where F(ON) is limit, but of cofinality $<$ON, kind 1 to the case F(ON) is successor, and kind Ω to the case F(ON) limit of cofinality ON.

If one says that F is a predecessor of $F + F'$, when $F' \neq \underset{\sim}{0}$, then this is sufficient to handle the predecessor relation in all cases but the case of kind Ω. It remains to find the predecessors of a dilator of kind Ω. This is done by means of the *separation of variables:* let $F = (1 + \mathrm{Id})^{\mathrm{Id}}$, i.e., $F(x) = (1 + x)^x$; then the idea is to 'separate the variables', in order to obtain the two variables functor $(1 + x)^y$: if one fixes the value of y, say a, then one obtains functors $(1 + \mathrm{Id})^a$ (i.e., $F^a(x) = (1 + x)^a$), and these functors are, by definition the predecessors of F. Separation of variables is detailed in Section 0.2.4; let us observe here that the predecessor relation is well founded, and that the predecessors of a given dilator have order type F(ON).

The principle of induction on dilators is a more sympathetic reformulation of Bar-induction of type 2: Bar-induction of type 2 is induction on well-founded trees with full type 2 branchings (equivalently full ordinal branchings); but whereas in the case of Bar-induction of type 2, the structure of the branchings is quite inexistant, the requirement that these branchings are given by a dilator permits one to understand really what is going on.

0.2.4. *Separation of variables*

If one wants to associate to a dilator of kind Ω a two-variable functor, then one must ask for a certain asymmetry between the variables: the requirement will be that the two-variable functor *does* depend on y, and, as a functor of y only, is a flower, i.e., naively speaking, $F(x, y)$ is a normal function of y (*bilators*).

It suffices to define $F(x, y)$ when F is perfect $\neq 1$, for, if $F = F' + F''$, one can define $F(x, y) = F'(x) + F''(x, y)$. In that case, the separation is defined by means of the permutation associated with F-denotations: in $(z_0; a_0, \ldots, a_{n-1}; a)$, let $i = \sigma_{z_0,n}(0)$; then the parameters $a_0, \ldots, a_i$ are y-wise, whereas $a_{i+1}, \ldots, a_{n-1}$ are x-wise.

What makes SEP (the functor of separation) an useful tool is the existence of an inverse functor UN (unification); by the way observe that the obvious way of unifying two-variables functors, i.e., the diagonal functor, is not invertible, so unification is not given by $F(x, x)$. (And, so, in Section 0.2.3, the separation is not $(1+x)^y$, but something close to it; however, when composed with the functor $\underset{\sim}{\omega}^{(1+\mathrm{Id})}$, unification coincides with diagonalisation.)

0.2.5. Recursion on dilators

To a principle of induction on dilators corresponds a similar principle of definition by recursion. This principle, which corresponds to a civilized version of Bar-recursion of type 2, is used in Section 5, to define a functor $\bigwedge$ from dilators to bilators (or equivalently to dilators of kind Ω), which corresponds to the iteration of composition along $F(\mathrm{ON})$, i.e., $\bigwedge F$ is the sum, iterated 'F times'.

There are many ways of doing so, and our choice is not the simplest, but we think that it is the more natural and elegant; previously circulated papers used easier constructions of $\bigwedge$, based on the fixed-point operation. The role of $\bigwedge$ will be clear only in part II, when we shall use it to measure cut-elimination bounds.

In order to define $\bigwedge$, we need first to define generalized (semi-)products of regular bilators; for instance if F is regular flower, then the product of ω copies of F is a functor F', such that $F'(x)$ enumerates the fixed points of F. $\bigwedge$ is defined to be a functor which transforms sums into products, i.e., $\bigwedge$ is a sort of exponential:

$$\bigwedge\Big(\sum_{i<x} F_i\Big)=\prod_{i<x} \bigwedge F_i\,.$$

0.2.6. Classification of natural transformations

Just let us say a word about the classification of a natural transformation: if $T\in I(F, F')$, then say that T is *deficient* if $T=T'+T''$, and $T''\in I(\underset{\sim}{0}, F'')$, for some $F''\neq\underset{\sim}{0}$; when T is not deficient, then F and F' are necessarily of the same kind, which is by definition the kind of T: so T can be of *five* kinds: deficient, 0, 1, ω, and Ω.

0.2.7. Continuity

A dilator induces a function from ON to ON; this function is continuous w.r.t. direct limits, but not in general w.r.t. the familiar topology of the ordinals (except when the dilator is a flower).

Let us first say frankly that the usual ordinal continuity is not at all a deep property of a function: obviously the function can do absolutely what it wants on non-limit values: all its sins will be forgiven provided it behaves well at limits; the best christians are not those who go to church weekly, but those whose everyday behaviour is beyond reproach: dilators do not necessarily go to church at limit points, but their behaviour is perfectly regular, and always obeys the same laws. For instance, $F(\omega+1)=\underrightarrow{\lim}^*(F(n+1), F(E_n\omega+E_1))$ is a continuity property which has absolutely no equivalent in terms of ordinal topology.

However, dilators can be represented as type 2 functionals if $X\subset\omega$ is the characteristic function of a well-ordered set, then $F(X)$ will be the characteristic function of its image under F; such a functional is obviously a continuous functional, with the familiar meaning of this expression, and commutation to direct limits expresses precisely that this functional is continuous. Conversely, a functional mapping characteristic functions of ordinals into themselves is not necessarily induced by a dilator, for instance, X and Y may be isomorphic, but not $F(X)$ and $F(Y)$.

One of the main advantages of the viewpoint of direct limits compared with the viewpoint of toplogy, is the extreme simplicity of the problem of limits in function spaces: in topology, one must distinguish carefully between pointwise and uniform convergence,... In the case of dilators, everything is so simple: in order to compute a direct limit of dilators, it suffices to do it pointwisely: if the pointwise limits exist, then the system has a limit...

0.3. An intrinsic theory of trees

0.3.1. Dendroids

There is no way of assigning trees (or equivalently fundamental sequences) to denumerable ordinals in an intrinsic way. The situation is radically different with dilators: there is a tree-like concept (*strongly homogeneous dendroids*), which is equivalent with dilators. As a consequence, dilators induce sh. dendroids and conversely: it is therefore possible to assign dendroids to dilators in an intrinsic way.

The natural idea is that of a *quasi-dendroid*: a tree with two possible kinds of branchings:

–ordinary ordinal branchings;

–branchings made up of underlined ordinals; these ordinals must be less than an ordinal fixed in advance, the *type* of D. The dynamic aspect of the theory lies in the underlined branchings.

When D is a quasi-dendroid of type y, and $f \in I(x, y)$, then it is possible to define the *multilation* of D:

–remove all sequences containing an underlined element $\underline{z}$, with $z \notin \mathrm{rg}(f)$,

–then, replace all $\underline{f(z)}$ by $\underline{z}$.

The result of this process is called ${}^fD$; it is a quasi-dendroid of type x. We shall say D is *homogeneous* when, roughly speaking, ${}^fD$ depends only on x (and not on $f \in I(x, y)$). In fact, the (main) condition is ${}^fD \asymp {}^{f'}D$, when $f, f' \in I(x, y)$, and $\asymp$ is an equivalence relation between quasi-dendroids, whose equivalence classes are called *dendroids*. (Another way of defining dendroids is to ask drastic conditions on quasi-dendroids, in such a way that each equivalence class contains exactly one point enjoying these conditions; this is the viewpoint of Section 6.)

A dendroid of type ω is *strongly homogeneous* iff for all $x \geqslant \omega$ there is an homogeneous quasi-dendroid $D^0(x)$ such that

$$D \asymp {}^fD^0(x) \quad \text{for all } f \in I(\omega, x).$$

A sh. dendroid induces a dilator LIN(D), simply by

$$\mathrm{LIN}(D)(x) = \text{order type of } D^0(x).$$

It is more delicate to prove that the functor LIN (*linearization*) is inversible; its inverse BCH (*branching*) can be constructed by means of the fine structure of dilators, i.e., of permutations and the § function. For instance, the underlined elements correspond to the parameters $x_0, \ldots, x_{n-1}$, in the denotations, and their order of appearance in the sequence is given by the permutation: $\underline{x}_{\sigma(0)}$ occurs first...

0.3.2. Hierarchies

Tree-like structures permit us to define various hierarchies; when D is a quasi-dendroid of type ω, such that all its non-underlined branchings are finite, then it is possible to define hierarchies: $\gamma_{D,s}$ of functions from N to N, and $\vartheta_{D,s}$ of functions from $N \times N$ to N:

(i) If $s \notin D^* = \{(\ )\} \cup \{t; \exists u(t * u \in D)\}$, then

$$\gamma_{D,s}(n) = 0, \qquad \vartheta_{D,s}(m, n) = n.$$

(ii) If $s \in D$, then

$$\gamma_{D,s}(n) = 1, \qquad \vartheta_{D,s}(m, n) = m + n.$$

(iii) If $s \in D^* - D$, then define $s_0, \ldots, s_{p-1}$ to be:
$-s * (\underline{0}), \ldots, s * (\underline{n-1})$ if s is followed by an underlined branching;
$-s * (i_0), \ldots, s * (i_{p-1})$, where $i_0, \ldots, i_{p-1}$ is the non-underlined branching starting from s otherwise; then, in both cases:

$$\gamma_{D,s}(n) = \gamma_{D,s_0}(n) + \cdots + \gamma_{D,s_{p-1}}(n),$$
$$\vartheta_{D,s}(m, n) = \vartheta_{D,s_0}(m, \vartheta_{D,s_1}(m, \ldots, \vartheta_{D,s_{p-1}}(m, n)\ldots)).$$

γ is the *pointwise* hierarchy.

It is easy to see that, provided $D = \mathrm{BCH}(F)$, (with $\gamma_D = \gamma_{D,()}$, $\vartheta_D = \vartheta_{D,()}$)

$$\gamma_D(n) = F(n), \qquad \vartheta_D(m, n) = (\bigwedge F)(m, n),$$

hence, the relation between the two hierarchies is easy to establish: if $D' = \mathrm{BCH}(\bigwedge F)$, then

$$\gamma_{D'}(n) \leq \vartheta_D(n, n) \quad \text{and} \quad \vartheta_D(m, n) \leq \gamma_{D'}(m + n).$$

0.3.3. Rungs and ladders

Instead of putting the viewpoint of trees at the first place, one can take the viewpoint of fundamental sequences; we did this in previous versions, and it is still possible to do it in a way that this approach is completely equivalent to dendroids or dilators. However, the difference of this approach with the approach of dendroids is not very significant, so we think that it is more reasonable to develop this concept at another place. But the very first concept of ladders was used to give the original proof of the hierarchy theorem, and, since this proof was never published, we think it may be of interest to include it here as an appendix; of course this original proof is simpler, and this greater simplicity is enhanced by the fact that our redaction is not at all directed towards the exposition of the hierarchy theorem.

0.3.4. Gardens

Another alternative direction is *gardens*; the concept of a garden was used in unpublished manuscripts of 76–77 (also: Oxford conference, 1976); the terrible complexity of everything connected with gardens made it necessary to look for a simpler viewpoint. However, in some cases, gardens may be of great interest: let us say shortly that in a garden, fundamental sequences are replaced by 'fundamental flowers'. The reader will find basic information on gardens in [10, 18]. (This concept turns out to be equivalent with dilators.) The relation of $\bigwedge$ to the Bachmann hierarchy is investigated in these papers.

0.3.5. Multi-dendroids

Dendroids represent an attempt to hide the category theoretic aspect of Π^1_2-logic inside the tree structure. But this attempt is not successful, because natural transformations are not encoded by the trees. So, in order to have a treatment of Π^1_2-logic completely free from categories, one introduces *multi-dendroids*, where ordinals are coloured in several colours, the idea being that the extra colours represent natural transformations (i.e., mutilations w.r.t. the non-homogeneous colours induce natural transformations of the functors corresponding to the homogeneous colours). Only a lack of space, and perhaps the fear of making overrepetitive proofs are the reasons why the viewpoint of multi-dendroids is not systematically developed. If another exposition of this work is written in the future, then the obvious thing to do is to start with multi-dendroids.

0.3.6. Pointwise and global constructions

If one defines the sum of two dendroids, then this operation induces a corresponding operation on the functors D^0: $(D+D')^0(x) = D^0(x) + D'^0(x)$; such an operation is *pointwise* (or local), because it is defined only by means of the values $D^0(x)$: the functor plays no role; in the definition of the hierarchy γ, the situation is the same. On the other hand, if one defines the composition of sh. dendroids, then it is not possible to express it as a pointwise construction: in order to have $(D \circ D')^0(x)$, we need $D'^0(x)$, but also $D^0(\underline{h}(D'^0(x))$: essentially the 'totality' of the functors is involved in the construction of the composition, even if one is interested in a specific point. Typically, the hierarchy λ and the functor $\bigwedge$ are of that sort: *global* constructions. Analyzing further, one finds that, in order to define $(D \circ D')$, one needs to 'blow up' D to the size $\underline{h}(D')$; this blowing up is perfectly determined by $D = D^0(\omega)$, but the fact that the result of this blowing up is still a well-founded tree cannot be predicted if one only knows that D is homogeneous. In fact, there is a complete change of viewpoint, concerning hierarchies:

–traditionally, it is not too 'incorrect' to think of a hierarchy as being indexed by an ordinal, i.e., in many cases the extra-structure plays a very minor role; for instance, if one wants to connect two variants of λ with each other, then the relation will be expressed by a function which already makes sense on ordinals, for instance an exponential.

–but the relation between γ and λ is of a distinct nature: if one knows the height of D, then one has no idea of the height of $\bigwedge(D)$.

Perhaps this means that, in fact, what is important is not the index D of λ_D, but really the function λ_D itself, viewed as a functor (i.e., a dilator); the ordinal $\underline{h}(D)$ appears now as the particular value $F(\omega)$ of the functor. Perhaps a good part of Π^1_2-logic can be explained by the replacement: hierarchies $\to$ dilators.

0.4. *Open questions*

The theory exposed in part I leaves relatively few open problems, especially if one compares it to the problems arising from part II. However there are some questions of (unequal) interest.

0.4.1. *Improving the concept of dendroid*

The concept of dendroid is satisfactory, because of its extreme simplicity. However, some unsympathetic phenomena arise: for instance the fact that LH does not commute to pull-backs, or the fact that SEP(F) is not in general a regular bilator. The question is: find a concept of *strong regularity* stable by predecessor, and usual operations (eventually slightly modify $\bigwedge$), and which implies regularity. The best would be that strongly regular dilators appear as functors from ON to itself preserving some extra structure, or even dilators that can be extended into a greater category ON′. (In terms of multi-dendroids, one can define D s.r. to mean that $s*(0)\in D^* \to s*(0)\in D$, for all s; unfortunately, this property is not easy to handle in terms of dilators.) One of the advantages of a solution to the problem of strong regularity is that the concept of a recursive dilator is not completely satisfactory from the effective viewpoint, while strongly regular and recursive dilators would be perfectly effective objects. (For instance the classification of dilators in four kinds is not effective, ...)[1]

0.4.2. *Comparison of dilators*

The predecessor relation between dilators is not a linear order, for instance Id and 10^{Id} are not comparable. The question is: How can one make the order total? Linear preorders are allowed, i.e., one can imagine to identify dilators. Of course, trivial solutions exist (compare F and F' on ω, for instance) and must be avoided. A good solution must have some meaning in terms of the 'ordinal classes' F(ON):

[1] *Added in proof* (Feb. '82): Recent work by Daniel Boquin seems to solve this question.

presumably, the solution, if it exists, will be connected to large cardinal axioms, and/or determinacy hypotheses.

0.4.3. Extension to higher types

The theory of dilators can be extended to finite types: this offers no essential difficulty. This extension would be very boring in the framework of this paper, but it will be done very soon elsewhere, because many essential results depend on it.

0.5. Sources

Essentially, the material exposed in this text is new. Of course, there are a certain number of category-theoretic trivialities, which cannot be considered as new things. Before this work, objects like dilators were only considered (as far as I know) in Peter Aczel's Ph.D.: the results are summarized in a short abstract [1]: he considers what I would now call regular flowers (but without commutation to &), and he proves (by methods not indicated in the abstract) that, if F is a normal functor, then one can make a new normal functor F', such $F'(x)$ is the xth fixed point of F. (If F is a regular flower, then it is possible to define F', for instance by taking the product of ω copies of F; but I do not know Aczel's original construction.)

The hierarchy theorem was announced in the introduction of [7], but it is only for the Oberwolfach meeting that I wrote the proof, in April 78; this proof (slightly modified) has been reproduced here in an appendix. I gave many different versions, corresponding to the various concepts I used in previous manuscripts of this text. Nowadays, many other proofs are available: [11, 16, 2, 4, 5, 15].

But the most independent contribution to Π^1_2-logic at this time is perhaps Herman Jervell's 'homogeneous trees', a concept he introduced in 79 [11], in order to present an alternative approach to Π^1_2-logic, essentially to replace the concepts of rungs and ladders. Masseron [14] has shown this concept to be equivalent to the concept of ladders. I decided finally to write this text with the concept of dilator (which is more general, and non-equivalent), but I tried to keep what could be kept of Jervell's ideas, and to adopt a Jervellian terminology in the sections on dendroids. (Dendroids are the concept which generalizes the concept of a Jervell tree.) I tried to transfer, when possible good ideas that where developed for homogeneous trees and/or ladders: for instance the operation $\sum^*$ of Section 7 was introduced by Marcel Masseron in [13] as an operation on rungs. Gandy [6] proposed another treelike interpretation of ladders; rungs and ladders were investigated in details by Khabaza [12].

What can be saved from the versions of 76–77 (gardens) is included in the two works in collaboration with Jacqueline Vauzeilles [9, 10], Jacqueline Vauzeilles made a third paper [18], proving the equivalence of gardens with dilators.

A certain number of contributions to Π^1_2-logic are not listed here: it is simply that they are connected with part II.

1. The category ON of ordinals

In this section, we shall introduce the main category-theoretic tools that will be used in the sequel. We assume that the reader knows the definition of *category, functor, natural transformation*. Other category-theoretic concepts will be explicitely defined. The ordinals are always considered (as usual from set theory) as the set of their predecessors.

1.1. Basic categories

Definition 1.1.1. If x and y are two linear orders, then $I(x, y)$ will be the set of strictly increasing mappings from x to y.

Definition 1.1.2. We define the following categories by their objects (in all these categories, the morphisms are given by $I(x, y)$):

(i) the category OL of linear orders; the objects are linear orders;
(ii) the category ON of ordinals; the objects are ordinals;
(iii) if x is an ordinal, the category ON$<x$; the objects are ordinals$<x$;
(iv) if x is an ordinal, the category ON$\leq x$; the objects are ordinals$\leq x$.

Definition 1.1.3. If x and y are ordinals such that $x \leq y$, then one defines $E_{xy} \in I(x, y)$ by: $E_{xy}(z) = z$; E_{xx} is abbreviated in E_x.

1.2. Some functors

Definition 1.2.1. We define the following functors from ON^2 into ON (let x, x', y, y' be ordinals, $f \in I(x, x')$, $g \in I(y, y')$)

(i) the functor *sum*:
$-x+y$ is the familiar ordinal sum of x and y;
$-$if $z<x$, then $(f+g)(z)=f(z)$, if $z<y$, then $(f+g)(x+z)=x'+g(z)$.

(ii) the functor *product*:
$-x \cdot y$ is the usual ordinal product of x and y;
$-$if $t<x$, $u<y$, then $(f \cdot g)(x \cdot u+t)=x' \cdot g(u)+f(t)$.

(iii) the functor *exponential*:
$-(1+x)^y$ is defined as usual;
$-(1+f)^g((1+x)^{u_1} \cdot (1+t_1)+\cdots+(1+x)^{u_n} \cdot (1+t_n))=(1+x')^{g(u_1)} \cdot (1+f(t_1))+\cdots+(1+x')^{g(u_n)} \cdot (1+f(t_n))$ for all $u_1, \ldots, u_n$, $t_1, \ldots, t_n$ such that $y>u_1>\cdots>u_n$, and $t_1, \ldots, t_n<x$.

Remark 1.2.2. It is not difficult in fact to extend these functors in such a way that they map OL^2 into OL; for instance one has to consider the sum of linear orders, ... Anyway, we don't have to bother too much about this extension, because direct limits give a general way of extending functors from ON into ON (or ON^2 into ON) into functors from OL into OL (or OL^2 into OL).

Definition 1.2.3. If x is an ordinal, define $\hat{x} = x+1$; if x, y are ordinals, and $f \in I(x, y)$, define $\hat{f} \in I(\hat{x}, \hat{y})$ by:

$$\hat{f}(z) = \sup_{t<z}(f(t)+1)).$$

(equivalently, we have: $\hat{f}(0) = 0$, $\hat{f}(z+1) = f(z)+1$, and, for z limit: $\hat{f}(z) = \sup_{t<z}(f(t))$.

Proposition 1.2.4. $\hat{}$ *is a functor from* ON *into itself.*

Proof. Left to the reader.

Definition 1.2.5. If x is a linear order, define $\mathrm{Opp}(x)$, to be the opposite order relation (same underlying set, order reversed); if $f \in I(x, y)$, define $\mathrm{Opp}(f) = f$.

Remark 1.2.6. When we have a functor from OL into OL, which has the property that $F(x)$ is a well-order for all $x \in \mathrm{ON}$, then one can consider its 'restriction to ON', which is defined by: $G(x)$ = unique ordinal isomorphic to $F(x)$, the functions $G(f) \in I(G(x), G(y))$ are such that $\phi_y F(f) = G(f)\phi_x$ (ϕ_x is the unique isomorphism between $F(x)$ and $G(x)$). But the functor *Opp* cannot be restricted into a functor from ON into itself, because the order opposite to an infinite well-order is not a well-order. However, the opposite of a finite linear order is again a finite linear order, and this means that *Opp* can be restricted into a functor from $\mathrm{ON}{<}\omega$ into itself; we have:

$$\widetilde{Opp}(n) = n \quad \text{and} \quad \widetilde{Opp}(f)(n-1-z) = m-1-f(z) \quad \text{if} \quad f \in I(n, m), \quad z < n.$$

1.3. Direct limits

Definition 1.3.1. Let $\mathscr{C}$ be a category, and let I be a non void ordered set; we shall always assume that I is *directed*, i.e., for all $i, j \in I$, there is $k \in I$ such that $i, j < k$.

A *direct system in* $\mathscr{C}$, *indexed by* I, appears as a family $(x_i, f_{ij})_{i,j \in I}$, such that:
(i) for all $i \in I$, x_i is an object of $\mathscr{C}$;
(ii) for all $i, j \in I$ such that $i < j$, f_{ij} is a $\mathscr{C}$-morphism from x_i to x_j;
(iii) for all $i \in I$, f_{ii} is the identity of x_i;
(iv) for all i, j, k in I such that $i < j < k$, $f_{ik} = f_{jk} f_{ij}$.

Definition 1.3.2. (i) A *direct system of morphisms* (with associated function φ) between the direct systems (x_i, f_{ij}) (indexed by I) and (y_l, g_{lm}) (indexed by L) is a family $(h_i)_{i\in I}$ of $\mathscr{C}$-morphisms from x_i to $y_{\varphi(i)}$ such that: φ is an increasing function from I to L, and for all $i, j \in I$:

$$i \prec j \rightarrow h_j f_{ij} = g_{\varphi(i)\varphi(j)} h_i.$$

(ii) If $(k_l)_{l\in L}$ is another direct system of morphisms (with associated function ψ) between (y_l, g_{lm}) and (z_p, d_{pq}) (indexed by P), then the *composition* of (k_l) and (h_i), $(h_i') = (k_l)(h_i)$ is defined by: $h_i' = k_{\varphi(i)} h_i$; it is immediate that (h_i') is a direct system of morphisms between (x_i, f_{ij}) and (z_p, d_{pq}) (with associated function $\psi\varphi$).

Definition 1.3.3. Let (x_i, f_{ij}) be a direct system in $\mathscr{C}$, indexed by I; the family $(x, f_i)_{i\in I}$ is said to be a *direct limit* of (x_i, f_{ij}) iff (i)–(iv) hold:

(i) x is an object of $\mathscr{C}$,

(ii) for all $i \in I$, f_i is a morphism from x_i to x;

(iii) for all $i, j \in I$ such that $i \prec j$, $f_i = f_j f_{ij}$;

(iv) if $(y, g_i)_{i\in I}$ is any family which satisfies (i)–(iii), then there is an *unique* morphism h from x to y such that, for all $i \in I$: $g_i = h f_i$ (Fig. 1).

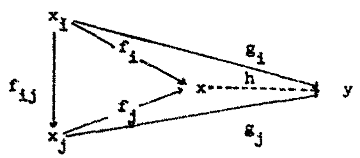


Fig. 1.

Remark 1.3.4. Conditions (i)–(iii) mean that it is possible to extend the given direct system into a system indexed by $I^* = I +$ a topmost element ∞; the extension is defined by: $x_\infty = x$, $f_{i\infty} = f_i$, $f_{\infty\infty} =$ identity of x_∞.

Remark 1.3.5. In fact, direct limits are defined in the more general context of an arbitrary non void ordered set, non necessarily directed; this generalization offers absolutely no interest for us; but one can consider direct systems indexed by non void directed preorders, without any significant difference with our definition.

Examples 1.3.6. (i) In the categories of Definition 1.1.2, there is a very simple way of reformulating Definition 1.3.3(iv):

(iv)′ $x = \bigcup_{i\in I} \mathrm{rg}(f_i)$, i.e., every point in x is in the range of some f_i.

(*Proof.* (iv)$\rightarrow$(iv)′: Let $X = \bigcup_{i\in I} \mathrm{rg}(f_i)$; define y and $k \in I(y, x)$ by $\mathrm{rg}(k) = X$, and $g_i \in I(x_i, y)$ by $f_i = k g_i$; since (y, g_i) enjoys (i), (ii), (iii), condition (iv) ensures the existence of $h \in I(x, y)$ such that $g_i = h f_i$, hence $f_i = k h f_i$, for all $i \in I$; from this, $kh(z) = z$ for all $z \in X$; so h maps X onto y, and since h is strictly increasing, this forces $X = x$.

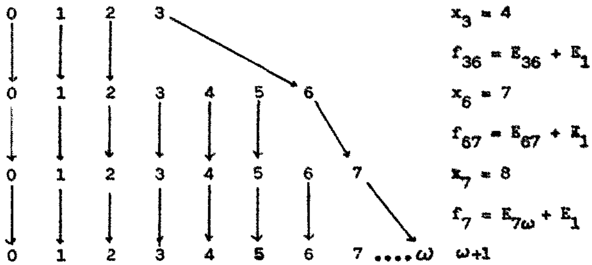


Fig. 2.

(iv)' → (iv): Let $z \in x$; we define $h(z)$ as follows: choose $i \in I$, and $z_i \in x_i$ such that $z = f_i(z_i)$ (consequence of (iv)'), and let $h(z) = g_i(z_i)$; this definition is possible: if $z = f_j(z_j)$, let k be $> i, j$; then $g_i(z_i) = g_k(f_{ik}(z_i)) = g_k(f_{jk}(z_j)) = g_j(z_j)$, using the fact that $f_{ik}(z_i) = f_{jk}(z_j)$, which is obvious from $z = f_k(f_{ik}(z_i)) = f_k(f_{jk}(z_j))$. h is strictly increasing: if $z < Z'$, one can choose $i \in I$, z_i, z'_i, with $z = f_i(z_i)$, $z' = f_i(z'_i)$, because I is directed; since $z_i < z'_i$, one gets $h(z) < h(z')$. By construction, $g_i(z_i) = h(f_i(z_i))$, so $g_i = hf_i$. The unicity of h is obvious.)

(ii) The simplest case of direct limit is that of a supremum: if x is a limit ordinal, let $I = x$, let $a_y = y$, and if $y < z < x$, let $f_{yz} = E_{yz}$; then (x, E_{yx}) is a direct limit of (a_y, f_{yz}); in that case the direct limit equals the sup (we have: x limit $\leftrightarrow (x, E_{yx}) = \varinjlim_x (y, E_{yy'})$).

(iii) If the system (x_i, f_{ij}) is such that all x_i's are equal to some fixed integer n, then the system has a direct limit (n, f_i) (hint: all functions f_{ij} are isomorphisms). But if x_i is contantly equal to some infinite ordinal, nothing can be said concerning an eventual direct limit. For instance, any denumerable limit ordinal x can be obtained as a directed limit of a system (x_i, f_{ij}), with all x's equal to ω.

(iv) We give now the crucial example of a system (x_i, f_{ij}), with the x_i's finite, and with a limit $(\omega + 1, f_i)$: I will be the integers, if $n \in I$, let $x_n = n + 1$, if $n \le m$, let $f_{nm} = E_{nm} + E_1$, i.e., $f_{nm}(z) = z$ if $z < n$, and $f_{nm}(n) = m$; define $f_n \in I(n+1, \omega+1)$ by $f_n = E_{n\omega} + E_1$, i.e., $f_n(z) = z$ if $z < n$, $f_n(n) = \omega$, then $(\omega + 1, f_n)$ enjoys obviously properties (i)–(iv)'. A picture may help (see Fig. 2).

(v) Similarily, the system $(n + n, E_{nm} + E_{nm})$ admits $(\omega + \omega, E_{n\omega} + E_{n\omega})$ as direct limit; the system $(n \cdot n, E_{nm} \cdot E_{nm})$ admits $(\omega \cdot \omega, E_{n\omega} + E_{n\omega})$ as direct limit; the system $((1+n)^n, (1 + E_{nm})^{E_{nm}}$ admits $(\omega^\omega, (1 + E_{n\omega})^{E_{n\omega}}$ as direct limit...

Proposition 1.3.7. (i) *If the direct limit of (x_i, f_{ij}) is (x, f_i), then x is unique up to isomorphism. In the category* ON *there is no isomorphism distinct from the identity, so in this category there is only one possible choice for the direct limit. (The fact that x is unique up to isomorphism explains the use of the article 'the': we speak of 'the' direct limit; another abuse of terminology is to refer to x as the direct limit of (x_i, f_{ij}), whereas the direct limit is (x, f_i)).*

(ii) *Supppose that* (x_i, f_{ij}) *and* (y_l, g_{lm}) *admit direct limits* (x, f_i) *and* (y, g_l), *and let* (h_i) *be a direct system of morphisms between* (x_i, f_{ij}) *and* (y_l, g_{lm}), *with associated function* φ, *then there is an unique morphism* h *from* x *to* y *such that, for all* $i \in I$: $hf_i = g_{\varphi(i)}h_i$; *this morphism is the 'direct limit of the* h_i*'s'. If* $(h'_i) = (k_l)(h_i)$, *and if* (z_p, d_{pq}) *as a direct limit, then the direct limits* h' *and* k *of* (h'_i) *and* (k_l) *satisfy:* $h' = kh$.

Proof. (i) If (x, f_i) and (y, g_i) are two direct limits of (x_i, f_{ij}), then by Definition 1.3.3(iv), there exist h and k such that $g_i = hf_i$ and $f_i = kg_i$ for all i; from this $f_i = khf_i$ for all i; Definition 1.3.3(iv) ensures that kh is the only morphism such that $f_i = khf_i$ for all i, so $kh = \text{identity}$, similarily $hk = \text{identity}$ (Fig. 3).

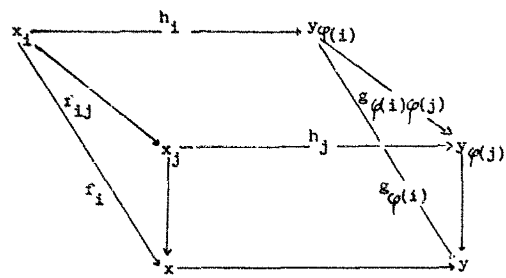


Fig. 3.

(ii) $(y, g_{\varphi(i)}h_i)$ enjoys Definition 1.3.3(i)–(iii); so there is an unique h from x to y such that $hf_i = g_{\varphi(i)}h_i$ for all i. The property of composition is immediate (Fig. 4).

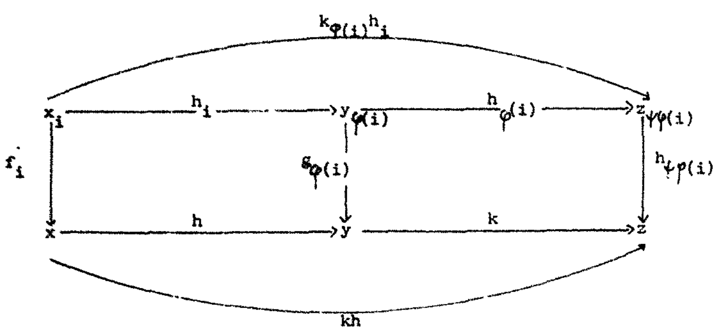


Fig. 4.

Examples 1.3.8. (i) Suppose that $(x_i)_{i\in I}$ and $(y_l)_{l\in L}$ are increasing families of ordinals, that φ is an increasing function from I to L, and that $f_i \in I(x_i, y_{\varphi(i)})$ are such that $i<j \to f_j$ extends f_i (i.e., if $z < x_i$ and $i < j$, then $f_i(z) = f_j(z)$); then it is

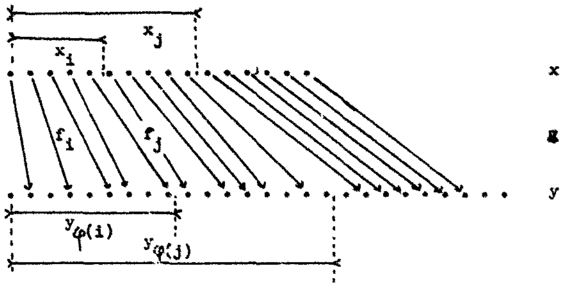


Fig. 5.

possible to define a function $g = \bigcup_i f_i$, $g \in I(\sup(x_i), \sup(y_l))$, by $g(z) = f_i(z)$, where i is any index such that $z < x_i$.

In fact, $\bigcup_i f_i$ is a very elementary case of direct limit of morphisms: recall (Example 1.3.6(ii)) that (with $x = \sup(x_i)$, $y = \sup(y_l)$) $(x, E_{x_i x})$ (resp. $(y, E_{y_l y})$) is the direct limit of the system $(x_i, E_{x_i x_j})$ of (resp. $(y_l, E_{y_l y_m})$); it is immediate that (f_i) is a direct system of morphisms between $(x_i, E_{x_i x_j})$ and $(y_l, E_{y_l y_m})$ (with associated function φ) and that $g = \bigcup_i f_i$ is the direct limit of the family (f_i) (Fig. 5).

(ii) If (x, f_i) is the direct limit of (x_i, f_{ij}), then, for all $i \in I$, f_i appears as a direct limit of morphisms: define I_i by: $I_i = \{l \in I;\ i \prec l\}$; then define the direct system of morphisms $(h_l)_{l \in I_i}$ (with associated function the identity function φ from I_i to I_i; between (y_l, g_{lm}) (with $y_l = x_i$, $g_{lm} = E_{x_i}$) and $(x_l, f_{lm})_{l,m} \in I_i$, by $h_l = f_{il}$. Then (y_l, g_{lm}) and (x_l, f_{lm}) admit (x_i, E_{x_i}) and (x_l, f_{lm}) respectively as direct limits, and f_i is the direct limit of (h_1) (immediate) (Fig. 6).

Theorem 1.3.9. *In the categories* ON, OL, *every object is a direct limit of integers, i.e., given any object x, then one can find a direct system (x_i, f_{ij}) and a family (f_i) such that (x, f_i) is the direct limit of (x_i, f_{ij}).*

Proof. Define $I = \{i;\ i \subset x,\ i \text{ finite}\}$, and order I by inclusion; I is obviously directed. Define

$-x_i =$ number of elements of i;

$-f_{ij}(p) = q$ iff the pth element of i in increasing order is the qth element of j;

$-f_i(p) =$ the pth element of i.

$$\begin{tikzcd}
x_i \arrow[r, "E_{x_i}"] \arrow[d, "f_{ij}"'] & x_i \arrow[r, "E_{x_i}"] \arrow[d, "f_{ik}"] & x_i \arrow[d, "f_i"] \\
x_j \arrow[r, "f_{jk}"'] & x_k \arrow[r, "f_k"'] & x
\end{tikzcd}$$

Fig. 6.

Conditions 1.3.3(i)–(iv)′ are obviously satisfied (in fact f_{ij} is defined by $f_i = f_j f_{ij}$; if $z \in x$, then $z \in \mathrm{rg}(f_{(z)})$; finite subsets (and not only subsets consisting of one element) are needed in order to ensure directedness). So (x, f_i) is the direct limit of (x_i, f_{ij}).

Remark that, although the x_i's are small, the indexing set I may be very big.

Notations 1.3.10. (i) $(x, f_i) = \varinjlim_I(x_i, f_{ij})$ when (x, f_i) is the direct limit of (x_i, f_{ij}); usually the index set I will be omitted, so our current notation will be $(x, f_i) = \varinjlim(x_i, f_{ij})$; if we want to express x without the f_i's, then we shall use the notations $x = \varinjlim_I^*(x_i, f_{ij})$ or $x = \varinjlim^*(x_i, f_{ij})$.

(ii) $h = \varinjlim_I(h_i)$ or $h = \varinjlim(h_i)$ when h is the direct limit of the system (h_i); if $(x, f_i) = \varinjlim(x_i, f_{ij})$, then we shall use the notation $f_i = \varinjlim_j(f_{ij})$: this notation is justified by Example 1.3.8(ii).

Remarks 1.3.11. (i) If x is denumerable and limit, then $x = \varinjlim^*(x_i, f_{ij})$ for some system (x_i, f_{ij}), with $x_i = \omega$ for all i: simply consider all subsets i of x of order type ω, which are cofinal in x...

(ii) If $(x, f_i) = \varinjlim(x_i, f_i)$ in ON, with all the x_i's finite, then it is immediate that $(Opp(x), Opp(f_i)) = \varinjlim(x_i, Opp(f_{ij}))$ in OL; but, if x is infiite, then $Opp(x)$ is not a well-order: it follows from this (and Theorem 1.4.2 below) that a direct system in ON needs not to have a direct limit.

1.4. Existence of direct limits

Theorem 1.4.1. *In* OL, *all direct systems have direct limits.*

Proof. Let (x_i, f_{ij}) be a direct system in OL; let X be the disjoint union of the x_i's, and define a binary relation R on X by: $(a, i)R(b, j)$ iff there exists $k > i, j$ such that $f_{ik}(a) r_k f_{jk}(b)$, r_k being the order relation on x_k; R is obviously a preorder on X, and, since I is directed, R is a total preorder; if S is the equivalence associated with R, X/S is totally ordered by R/S; let us call this ordered set x, and define $f_i \in I(x_i, x)$ by $f_i(z)$ = equivalence class of (z, i) modulo S; then it is immediate that (x, f_i) enjoys Definition 1.3.3(i)–(iv)′, so we have

$$(x, f_i) = \varinjlim(x_i, f_{ij}).$$

Theorem 1.4.2. *Given a direct system (x_i, f_{ij}) in* ON, *let x be its limit, when this system is considered as a direct system in* OL; *then (x_i, f_{ij}) admits a direct limit in* ON *iff x is a well-order; furthermore, when the limit exists in* ON, *then it is the (unique) ordinal isomorphic to x.*

Proof. We shall assume that (in OL) $(x, f_i) = \varinjlim(x_i, f_{ij})$;

(i) If x is well-ordered, we may assume as well that x is an ordinal (by eventually replacing x by its isomorphic ordinal); but, if (x, f_i) is the direct limit of (x_i, f_{ij}) in OL, it is a fortiori the direct limit of (x_i, f_{ij}) in the subcategory ON of OL.

(ii) Conversely, suppose that (x_i, f_{ij}) admits a direct limit (y, g_i) in ON; (y, g_i) enjoys Definition 1.3.3(i)–(iii) in OL, hence there exists $h \in I(x, y)$ such that $g_i = hf_i$; but if $h \in I(x, y)$ and y is a well-order, so is x.

Theorem 1.4.3. *Let L, I be two directed ordered sets, and let φ be an increasing function from L to I; suppose that (x_i, f_{ij}) is a direct system in ON indexed by I, and define a new system indexed by L, (y_l, g_{lm}) by $y_l = x_{\varphi(l)}$, $g_{lm} = f_{\varphi(l)\varphi(m)}$; then:*

(i) *if (x_i, f_{ij}) admits a direct limit in ON, then (y_l, g_{lm}) admits a direct limit in ON.*

(ii) *If $\varphi(L)$ is cofinal in I, then the converse of* (i) *holds; furthermore, if $(x, f_i) = \varinjlim(x_i, f_{ij})$, then $(x, f_{\varphi(l)}) = \varinjlim(y_l, g_{lm}) = \varinjlim(x_{\varphi(l)}, f_{\varphi(l)\varphi(m)})$.*

Proof. Suppose that $\varphi(L)$ is cofinal in I: then (i) and (ii) are immediate, and true for an arbitrary category. So it is enough to treat the case where L is a subset of I, and φ is the canonical injection; we need only to prove (i): suppose that (x_i, f_{ij}) admits a limit along I, say (x, f_i), then the family $(x, f_i)_{i\in L}$ satisfies Definition 1.3.3(i)–(iii) w.r.t. $(x_i, f_{ij})_{i,j\in L}$; if $(y, g_i)_{i\in L}$ is the direct limit of this system in OL, then by Definition 1.3.3(iv) there exists $h \in I(y, x)$ such that $f_i = hg_i$, so the order y is a well-order, and by Theorem 1.4.2, $(x_i, f_{ij})_{i,j\in L}$ has a direct limit in ON.

Corollary 1.4.4. *Given a direct system (x_i, f_{ij}) in ON, and a family (x, f_i) in ON enjoying Definition* 1.3.3(i)–(iii), *then one can find a direct limit for (x_i, f_{ij}) in ON.*

Proof. This can easily be obtained from Theorem 1.4.2 or 1.4.3; for instance extend (x_i, f_{ij}) into a system indexed by $I^* = I +$ a topmost element ∞, as in Remark 1.3.4; then apply Theorem 1.4.2 to this system (which has obviously a direct limit, since it has a topmost element) and its restriction to I: (x_i, f_{ij}) has therefore a direct limit in ON. (Alternative method: consider the union of the ranges of the morphisms f_i, as in Example 1.3.6(i).)

Theorem 1.4.5. *Let $\mathscr{C}$ be one of the categories ON or OL, and let $(x_{il}, f_{il,jm})$ be a double inductive system indexed by a product $I \times L$, with product order; then*

(i) *If for each $i \in I$, the direct system (indexed by L) $(x_{il}, f_{il,im})$ has a direct limit, and if the system (indexed by I) $(\varinjlim^*_L(x_{il}, f_{il,im}), \varinjlim_L(f_{il,jl}))$ has a direct limit, then the full double system has a limit, and conversely.*

(ii) *If the full double system has a limit, then*

$$\varinjlim_{I\times L}(x_{il}, f_{il,jm}) = \left(\varinjlim_I{}^*\left(\varinjlim_L{}^*(x_{il}, f_{il,im}), \varinjlim_l(f_{il,jl})\right), \varinjlim_i\left(\varinjlim_m(f_{il,jm})\right)\right).$$

(iii) *If (h_{il}) is a direct system of morphism between $(x_{il}, f_{il,jm})$ and $(y_{il}, f'_{il,jm})$, and if these double systems have direct limits in $\mathscr{C}$, then*

$$\varinjlim_{I\times L}(h_{il}) = \varinjlim_{I}\Big(\varinjlim_{L}(h_{il})\Big).$$

Proof. (i) Assume that $(x_i, g_{il}) = \varinjlim_I (x_{il}, f_{il,im})$; define $L^* = L +$ a topmost element ∞; extend the system $(x_{il}, f_{il,jm})$ to $I\times L^*$ by means of:

$$x_{i\infty} = x_i, \qquad f_{i\infty,j\infty} = \varinjlim_{L}(f_{il,jl}), \qquad f_{il,j\infty} = g_{jl}f_{il,jl};$$

the extended system is easily seen to be a direct system. By hypothesis, the restriction of this sytem to the cofinal subset $I\times\{\infty\}$ has a direct limit (x, g_i), hence the full system (indexed by $I\times L^*$) has the direct limit $(x, g_i f_{il,i\infty})$. We show that the original system (indexed by $I\times L$) has the limit $(x, g_i g_{il})$: Properties 1.3.3(i)–(iii) are easily checked. Now, if (y, h_{il}) enjoys properties (i)–(iii), it is possible to extend this family to $I\times L^*$ by means of $h_i = \varinjlim_L(h_{il})$. Since this extended family still enjoys Definition 1.3.3(i)–(iii), there exists an unique h such that $h_{il} = hg_i f_{il,i\infty}$, for all $il\in I\times L^*$, so $h_{il} = hg_i g_{il}$ for all $il \in I\times L$; moreover, if h' is such that $h_{il} = h'g_i g_{il}$ for all $il\in I\times L$, then $h_{il} = h'g_i f_{il,i\infty}$ for all $il\in I\times L^*$: it suffices to look to the case $l\in L$: then $h'g_i g_{il} f_{il,i\infty} = h'g_i g_{il} f_{il,il} = h'g_i g_{il} = h_{il}$; from this it follows that $h' = h$. In fact, we have just proved formula (ii), since obviously:

$$x = \varinjlim_{I}{}^*(x_i, f_{i\infty,j\infty}) = \varinjlim_{I}{}^*\Big(\varinjlim_{L}{}^*(x_{il}, f_{il,im}), \varinjlim_{L}(f_{il,jl})\Big),$$

$$g_i g_{il} = \varinjlim_{i}\Big(\varinjlim_{m}(f_{im,jm})\Big)\circ\varinjlim_{m}(f_{il,im}) = \varinjlim_{j}\Big(\varinjlim_{m}(f_{il,jm})\Big).$$

Conversely, if the full double system has a limit, then the restricted system $(x_{il}, f_{il,im})$ has a limit in $\mathscr{C}$ (if $\mathscr{C} = \mathrm{OL}$, this is trivial (see Theorem 1.4.1), if $\mathscr{C} = \mathrm{ON}$, apply Theorem 1.4.3); if this limit is (x_i, g_{il}), then the system $(x_i, \varinjlim_L(f_{il,jl}))$ admits the direct limit $(x, g_i g_{il})$ in OL, by the above proof, and with the same notations; if $\mathscr{C} = \mathrm{OL}$, there is nothing to prove, if $\mathscr{C} = \mathrm{ON}$, observe that x is an ordinal by hypothesis, and apply Theorem 1.4.2 (Fig. 7).

(ii) Has been proved above.

(iii) If $h = \varinjlim_{I\times L}(h_{il})$, then h is uniquely determined by: $hg_i g_{il} = g'_i g'_{il} h_{il}$; but, if $h_i = \varinjlim_L(h_{il})$, then $h_i g_{il} = g'_{il} h_{il}$, and if $h' = \varinjlim_I(h_i)$, then $h'g_i = g'_i h_i$; so $h'g_i g_{il} = g'_i g'_{il} h_{il}$, so $h = h'$, and so $h = \varinjlim_{I\times L}(h_{il}) = \varinjlim_I(\varinjlim_L(h_{il}))$ (Fig. 8).

Corollary 1.4.6. *In* OL *or* ON, *if one of the expressions*

$$\varinjlim_{I}\Big(\varinjlim_{L}{}^*(x_{il}, f_{il,im}), \varinjlim_{L}(f_{il,jl})\Big) \quad\text{or}\quad \varinjlim_{L}\Big(\varinjlim_{I}{}^*(x_{il}, f_{il,jl}), \varinjlim_{I}(f_{il,im})\Big)$$

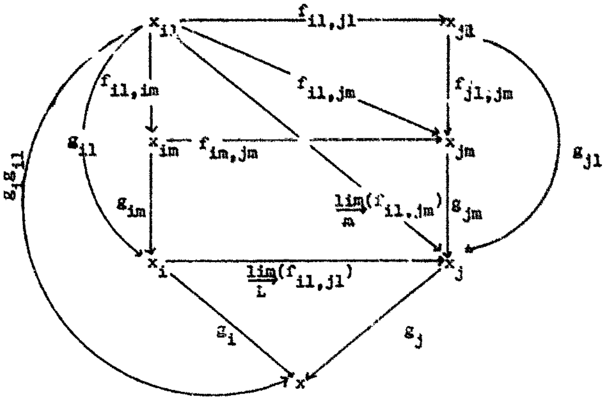


Fig. 7.

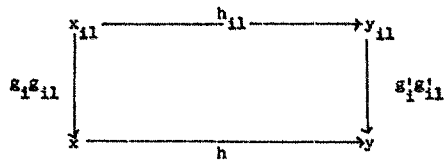


Fig. 8.

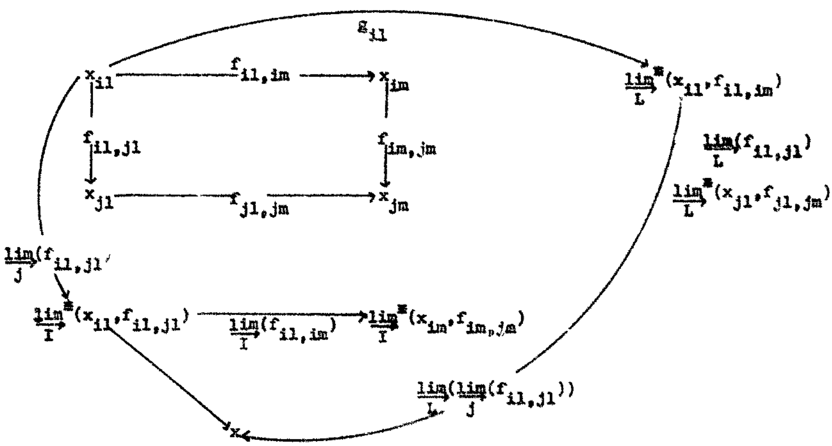


Fig. 9.

exists, then the other exists too; furthermore, we have:

$$\varinjlim_{I}\left(\varinjlim_{L}{}^{*}(x_{il}, f_{il,im}), \varinjlim_{L}(f_{il,jl})\right) =$$

$$\left(\varinjlim_{L}{}^{*}\left(\varinjlim_{I}{}^{*}(x_{il}, f_{il,jl}), \varinjlim_{I}(f_{il,im})\right), \varinjlim_{L}\left(\varinjlim_{I}(f_{il,jl})\right)\right) \quad \text{(Fig. 9)}.$$

1.5. Pull-backs

Definition 1.5.1. Let x_1, x_2, x_3 be objects of $\mathscr{C}$, let f_1, f_2, f_3 be morphisms from respectively x_1, x_2, x_3 to x; f_3 is said to be a *pull-back* of f_1 and f_2 iff

(i) there exist $\mathscr{C}$-morphisms f_{31} and f_{32} from x_3 to x_1 and x_2 such that

$$f_3 = f_1 f_{31} = f_2 f_{32};$$

(ii) given any other solution $(x'_3, f'_3, f'_{31}, f'_{32})$, then there exists a *unique* morphism h from x'_3 to x_3 such that $f'_{31} = f_{31}h$ and $f'_{32} = f_{32}h$ (Fig. 10).

$$\begin{tikzcd}
 & x_1 \arrow[dr, "f_1"] & \\
x'_3 \arrow[ur, "f'_{31}"] \arrow[r, "h"] \arrow[dr, "f'_{32}"'] & x_3 \arrow[u, "f_{31}"'] \arrow[d, "f_{32}"] \arrow[r, "f_3"] & x \\
 & x_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Fig. 10.

Notation 1.5.2. $f_3 = f_1 \,\&\, f_2$, when f_3 is a pull-back of f_1 and f_2.

Remark 1.5.3. Pull-backs are a very special case of inverse limit; the general concept of inverse limit is of no interest here.

Remark 1.5.4. 'The' pull-back is unique up to isomorphism: if f'_3 is another pull-back of f_1 and f_2, then there is an isomorphism h from the source x'_3 of f'_3 to x_3, such that $f'_3 = f_3 h$ (justification similar to Proposition 1.3.7(i)); in ON, pull-backs will therefore be uniquely determined.

Theorem 1.5.5. *In* ON *and* OL, *pull-backs always exist. In fact,* $f_3 = f_1 \,\&\, f_2$ *iff* $\mathrm{rg}(f_3) = \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$.

Proof. Any solution f_3 of Definition 1.5.1(i) enjoys $\mathrm{rg}(f_3) \subset \mathrm{rg}(f_i)$, for $i = 1, 2$, so $\mathrm{rg}(f_3) \subset \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$; conversely, if f_3 is such that $\mathrm{rg}(f_3) \subset \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$, then

Definition 1.5.1(i) holds: if $z \in x_3$, then $f_3(z) \in \mathrm{rg}(f_1)$, hence $f_3(z) = f_1(z')$ for some $z' \in x_1$, so one can put $f_{31}(z) = z'$.

If $\mathrm{rg}(f_3) = \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$, then given an arbitrary f_3' such that $\mathrm{rg}(f_3') \subset \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$, define $h \in I(x_3', x_3)$ by $h(z) =$ the unique z' such that $f_3'(z) = f_3(z')$; then it is immediate that $f_{31}' = f_{31}h$, $f_{32}' = f_{32}h$, and h is obviously uniquely determined. Conversely suppose that $f_3 = f_1 \,\&\, f_2$ and apply Definition 1.5.1(ii) to f_3' such that $\mathrm{rg}(f_3') = \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$, then $f_3' = f_3 h$, so $\mathrm{rg}(f_3) \supset \mathrm{rg}(f_3')$, hence $\mathrm{rg}(f_3) = \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$.

Theorem 1.5.6. *In* ON *or* OL, *let* (f_i^1), (f_i^2), (f_i^3) *be direct systems of morphisms between respectively* (x_i^1, g_{ij}^1), (x_i^2, g_{ij}^2), (x_i^3, g_{ij}^3) *and* (x_l, g_{lm}), *with the same associated function* φ, *and assume that* (x_l, g_{lm}) *has a direct limit; then*

$$\forall i \in I \quad f_i^3 = f_i^1 \,\&\, f_i^2 \to \varinjlim(f_i^3) = \varinjlim(f_i^1) \,\&\, \varinjlim(f_i^2).$$

Proof. First we treat the case where the index set L of (x_l, g_{lm}) consists of one element a; let $x = x_a$. Then we have to prove that

$$\bigcup_{i \in I} (\mathrm{rg}(f_i^1) \cap \mathrm{rg}(f_i^2)) = \Big(\bigcup_{i \in I} \mathrm{rg}(f_i^1)\Big) \cap \Big(\bigcup_{i \in I} \mathrm{rg}(f_i^2)\Big),$$

by double inclusion:

– the sense $\subset$ is obvious;

– conversely, if $a \in (\bigcup \mathrm{rg}(f_i^1)) \cap (\bigcup \mathrm{rg}(f_i^2))$, then, for some $i, j \in I$, $a \in \mathrm{rg}(f_i^1)$ and $a \in \mathrm{rg}(f_j^2)$; if $i, j < k$, then $a \in \mathrm{rg}(f_k^1) \cap \mathrm{rg}(f_k^2)$.

For the general case, let us observe, that, in ON and OL, $h(f \,\&\, g) = hf \,\&\, hg$; if $(x, g_l) = \varinjlim(x_l, g_{lm})$, then let $h_i^k = g_{\varphi(i)} f_i^k$; then $\varinjlim(f_i^3) = \varinjlim(h_i^3) = \varinjlim(h_i^1) \,\&\, \varinjlim(h_i^2) = \varinjlim(f_i^1) \,\&\, \varinjlim(f_i^2)$.

Remark 1.5.7. The similarity of the concepts of pull-back and intersection is striking: the pairs (x, X), when $X \subset x$ can be identified with morphisms of target X, by means of $X = \mathrm{rg}(f)$; if f, g are represented respectively by (x, X) and (y, Y), then $f \,\&\, g$ will be represented by $(x, X \cap Y)$; from this commutativity, associativity ... of pull-backs is obvious.

2. Elementary properties of dilators

In this section, we shall investigate the most obvious features of dilators; more subtle properties will be studied in the next section.

2.1. Commutation to direct limits

Definition 2.1.1. Let F be a functor from the category $\mathscr{C}$ to the category $\mathscr{D}$; F is said to *commute to direct limits* if the following holds: given any system (x_i, f_{ij})

with a direct limit (x, f_i) in $\mathscr{C}$, then the system $(F(x_i), F(f_{ij}))$ has the direct limit $(F(x), F(f_i))$ in $\mathscr{D}$.

Remark 2.1.2. If F is any functor from $\mathscr{C}$ to $\mathscr{D}$, and if $(x, f_i) = \varinjlim(x_i, f_{ij})$, then $(F(x), F(f_i))$ enjoys conditions 1.3.3(i)–(iii) w.r.t. the direct system $(F(x_i), F(f_{ij}))$; so, to say that F commutes to direct limits means that condition (iv) is fulfilled.

Examples 2.1.3. (i) The functors sum, product, exponential from ON^2 to ON commute to direct limits.

(ii) The functor Opp from OL to OL commutes to direct limits.

(iii) The functor $\hat{}$ from ON to ON does not commute to direct limits.

All these claims will be justified later in the section.

Theorem 2.1.4. *Suppose that the functor F from $\mathscr{C}$ to $\mathscr{D}$ commutes to direct limits; let (h_i) be a direct system of morphisms with a limit h in $\mathscr{C}$: then $(F(h_i))$ admits the limit $F(h)$ in $\mathscr{D}$.*

Proof. Suppose that (h_i) is a direct system of morphisms between (x_i, f_{ij}) (with limit (x, f_i)) and (y_i, g_{ij}) (with limit (y, g_i)); then $h = \varinjlim(h_i)$ is uniquely determined by the conditions $hf_i = g_ih_i$; by hypothesis $(F(x), F(f_i)) = \varinjlim(F(x_i), F(f_{ij}))$, $(F(y), F(g_i)) = \varinjlim(F(y_i), F(g_{ij}))$, and the direct limit k of $(F(h_i))$ is therefore uniquely determined by the conditions $kF(f_i) = F(g_i)F(h_i)$; but $F(h)$ satisfies these conditions, hence $\varinjlim(F(h_i)) = k = F(h) = F(\varinjlim(h_i))$.

Theorem 2.1.5. *Let $\mathscr{C}$ be a category closed under direct limits (i.e., in $\mathscr{C}$, all direct systems have direct limits, for instance $\mathscr{C} = \mathrm{OL}$), then*

(i) *If F is a functor from $\mathrm{ON}{<}\omega$ to $\mathscr{C}$, then F can be extended into a functor G from OL into $\mathscr{C}$, commuting to direct limits.*

(ii) *If F and H are functors from $\mathrm{ON}{<}\omega$ to $\mathscr{C}$, if T is a natural transformation from F to H, then if G and K are extensions of F and H into functors commuting to direct limits from OL to $\mathscr{C}$, then T can be extended uniquely into a natural transformation from G to K.*

Proof. (i) Using Theorem 1.3.9, construct a family (a_l, j_{lm}) indexed by the collection of finite subsets of linear orders, such that for any linear order x, $(x, h_l) = \varinjlim_{L(x)}(a_l, h_{lm})$, with $L(x) = \{i; i \subset x\}$; the orders a_l are integers. Define $G(x)$ by $G(x) = \varinjlim^*_{L(x)}(F(a_l, F(h_{lm}))$; one may choose $G(x) = F(x)$ when x is finite, because, in that case, $L(x)$ admits x as its topmost element, hence $\varinjlim^*_{L(x)}(F(a_l), F(h_{lm})) = F(a_x) = F(x)$. If x, y are linear orders and $f \in I(x, y)$, then f appears as the direct limit of a system (f_l) between $(a_l, h_{lm})_{l,m\in L(x)}$ and $(a_l, h_{lm})_{l,m\in L(y)}$, with associated function $\varphi(i) = f(i)$: $f_l = E_{a_l} = E_{a_{f(l)}}$.

Then define $G(f) \in I(G(x), G(y))$ by $G(f) = \varinjlim(F(f_l))$: it is immediate that G is a functor, i.e., $G(E_x) = E_{G(x)}$ and $G(fg) = G(f)G(g)$; furthermore, observe that,

$$\begin{tikzcd}
F(a_1) \arrow[r, "T(a_1)"] \arrow[d, "F(h_{1m})"'] \arrow[dd, bend right=60, "G(h_1)"'] & H(a_1) \arrow[d, "H(h_{1m})"] \arrow[dd, bend left=60, "K(h_1)"] \\
F(a_m) \arrow[r, "T(a_m)"'] & H(a_m) \\
G(x) \arrow[r, "U(x)"'] & K(x)
\end{tikzcd}$$

Fig. 11.

$(G(x), G(h_l)) = \varinjlim_{L(x)}(G(a_l), G(h_{lm}))$. It remains to prove that G commutes to direct limits: suppose that, in OL $(x, f_i) = \varinjlim_I(x_i, f_{ij})$. Define a new direct system $(y_{il}, g_{il,jm})$ indexed by $I \times L(x)$ as follows: if $\varphi(il) = \mathrm{rg}(f_i) \cap l$, let $y_{il} = a_{\varphi(il)}$, $g_{il,jm} = h_{\varphi(il),\varphi(jm)}$, and let $(z_{il}, k_{il,jm})$ be the image of this system under F; then we have:

$$\varinjlim_{I \times L(x)} (z_{il}, k_{il,jm}) = (G(x), G(g_{il}))$$

(immediate from the definition of G and from Theorem 1.4.3(ii)), but also

$$\begin{aligned}
\varinjlim_I (G(x_i), G(f_{ij})) &= \varinjlim_I \left(\varinjlim^*_{L(x)}(z_{il}, k_{il,im}), \varinjlim_{L(x)}(k_{il,jl}) \right) \\
&= \left(\varinjlim^*_{I \times L(x)} (z_{il}, k_{il,jm}), \varinjlim_{L(x)} \left(\varinjlim_i (k_{il,jl}) \right) \right);
\end{aligned}$$

the last equality follows from Corollary 1.4.6; since it is immediate that $\varinjlim_i(k_{il,jl}) = F(h_{\varphi(il)l})$ (because for sufficiently great j, $\varphi(jl) = l$), one concludes that $\varinjlim_{L(x)}(\varinjlim_i(k_{il,jl}))$ is equal to $G(f_i)$, hence

$$(G(x), G(f_i)) = \varinjlim_I (G(x_i), G(f_{ij})).$$

(ii) K is defined by $(K(x), K(h_l)) = \varinjlim_{L(x)}(H(a_1), H(h_{lm}))$, so, it is possible to define $U(x) \in I(G(x), K(x))$ by $U(x) = \varinjlim(T(g_l))$, (g_l) being the direct system between $(a_l, h_{lm})_{l,m \in L(x)}$ and itself (with associated function the identity) defined by $g_l = E_{a_l}$. The fact that U is a natural transformation is immediate (Fig. 11).

Remarks 2.1.6. (i) It is easy to extract the following remark from the proof of Theorem 2.1.5: all functors from $\mathrm{ON}{<}\omega$ to $\mathscr{C}$ commute to direct limits.

(ii) The functor G of Theorem 2.1.5(i) is not unique, because of the indetermination of the direct limit in $\mathscr{C}$; however one can remark that:
–If G' is another solution, then there exists an unique natural transformation T from G to G', which is the identity on the respective restrictions of G and G' to $\mathrm{ON}<\omega$ (apply Theorem 2.1.5(ii)); furthermore, this transformation T has an inverse T', i.e., $T'T$ is the identity of G, TT' is the identity of G' (obvious, left to the reader).
–Suppose that F maps $\mathrm{ON}<\omega$ into ON, and let G be an extension of F into a functor from OL to OL commuting to direct limits; suppose that G has the following property: if x is a well-order, so is $G(x)$, then it is possible to choose, when $x \in \mathrm{ON}$, $G(x)$ as an ordinal: hence G maps ON into ON: in this very special situation, we have extended F into a functor (the restriction of G to ON) from ON to ON commuting to direct limits: this extension is unique.
(iii) When specific extensions G and K have chosen, the extension U of T defined in Theorem 2.1.5(ii) is uniquely determined.

Definition 2.1.7. (i) If F, G are as in Theorem 2.1.5(i), then we shall say that G is *the extension of F by direct limits*. 'The' is an abuse of language, however, Remark 2.1.6(ii) above shows that F admits at most one extension by direct limits into a functor from ON to ON.
(ii) If F, G, H, K, T, U are as in Theorem 2.1.5(ii), then we shall say that U is *the extension of F by direct limits.*

Corollary 2.1.8. *Let F, G be functors from* ON *or* OL *into* $\mathscr{C}$; *then*
(i) *F commutes to direct limits iff it commutes to direct limits of the form* $\varinjlim(x_i, f_{ij})$, *with x_i finite for all $i \in I$.*
(ii) *F commutes to* $\varinjlim$ *iff: for all x, for all $z \in F(x)$, there exists an integer n and a function $f \in I(n, x)$ s.t. $z \in \mathrm{rg}(F(f))$.*
(iii) *If F and G are functors from* ON *to* ON *commuting to* $\varinjlim$, *then $F = G$ iff they coincide on* $\mathrm{ON}<\omega$.

Proof. (i) If F commutes to the limits $\varinjlim(x_i, f_{ij})$ with x_i finite, then F coincide with the extension of $F \restriction \mathrm{ON}<\omega$ built in the proof of Theorem 2.1.5, so F commutes to $\varinjlim$.
(ii) If F commutes to $\varinjlim$, write $(x, f_i) = \varinjlim(x_i, f_{ij})$, with x_i finite, so $(F(x), F(f_i)) = \varinjlim(F(x_i), F(f_{ij}))$; so, if $z \in F(x)$, $z \in \mathrm{rg}(F(f_i))$ for some i, take $n = x_i$, $f = f_i$. Conversely, assume that F has the property of (ii); if $(x, f_i) = \varinjlim(x_i, f_{ij})$, with all x_i finite, then $(F(x), F(f_i))$ enjoys Definition 1.3.3(i)–(iii) w.r.t. $(F(x_i), F(f_{ij}))$; if $z \in F(x)$, let n and $f \in I(n, x)$ such that $z \in \mathrm{rg}(F(f))$; if i is such that $\mathrm{rg}(f) \subset \mathrm{rg}(f_i$ (so $f = f_i k$), then $\mathrm{rg}(F(f)) \subset \mathrm{rg}(F(f_i))$ (because $F(f) = F(f_i)F(k)$), so $z \in \mathrm{rg}(F(f_i))$; this proves (iv)$'$, and apply (i) to conclude that F commutes to $\varinjlim$.
(iii) is immediate from Definition 2.1.7.

Remark 2.1.9. We have also to deal with binary functors, i.e., functors from for instance ON^2 to ON; we indicate briefly what we have to know, without justification:

(i) in the product category $\mathscr{C}\times\mathscr{D}$, we have:

$$((x, y), (f_i, g_i)) = \varinjlim((x_i, y_i), (f_{ij}, g_{ij})) \quad \text{iff}$$

$$(x, f_i) = \varinjlim(x_i, f_{ij}) \quad \text{and} \quad (y, g_i) = \varinjlim(y_i, g_{ij}).$$

(ii) the functor F from $\mathscr{C}\times\mathscr{D}$ into $\mathscr{E}$ commutes to direct limits iff:
–for any object x of $\mathscr{C}$, the functor G from $\mathscr{D}$ to $\mathscr{E}$ defined by $G(y) = F(x, y)$, $G(f) = F(\mathrm{id}_x, f)$ commutes to direct limits.
–for any object y of $\mathscr{D}$, the functor G from $\mathscr{C}$ to $\mathscr{E}$ defined by $G(x) = F(x, y)$, $G(f) = F(f, \mathrm{id}_y)$ commutes to direct limits.

(iii) Any functor F from $(\mathrm{ON}<\omega)^2$ to OL can be extended into a functor from OL^2 into OL, commuting to direct limits.

(iv) In order that F from ON^2 to ON commutes to direct limits it is necessary and sufficient that for all ordinals x, y and any $z < F(x, y)$, there exist integers n and m and morphisms $f \in I(n, x)$, $g \in I(m, y)$, such that $z \in \mathrm{rg}(F(f, g))$.

Examples 2.1.10. It is now possible to justify the claims of Examples 2.1.3:

(i) Take for instance the functor exponential; we use Remark 2.1.9(iv): take ordinals x, y and $z < (1+x)^y$; one can write $z = (1+x)^{u_1}\cdot(1+t_1)+\cdots+(1+x)^{u_n}(1+t_n)$ with $y > u_1 > \cdots > u_n$, and $t_1, \ldots, t_n < x$; let $A = \{t_1, \ldots, t_n\}$, $B = \{u_1, \ldots, u_n\}$ if $f \in I(p, x)$ and $g \in I(n, y)$ are such that $\mathrm{rg}(f) = A$, $\mathrm{rg}(g) = B$, then, obviously, $z \in \mathrm{rg}(F(f, g))$. So exp. commutes to direct limits.

(ii) Obvious from the definition of commutation to $\varinjlim$.

(iii) consider the functor $\underline{1}+\mathrm{Id}$ defined by $(\underline{1}+\mathrm{Id})(x) = 1+x$, $(\underline{1}+\mathrm{Id})(f) = E_1+f$; it is immediate that this functor commutes to direct limits (see Examples 2.1.11); but this functor coincides with $\hat{}$ on the category $\mathrm{ON}<\omega$; but $(\underline{1}+\mathrm{Id})(\omega) = \omega$, whereas $\hat{\omega} = \omega+1$, so by Corollary 2.1.8(iii), $\hat{}$ cannot commute to direct limits

Examples 2.1.11. (i) If x is an object of ON (or OL), then the functor $F(y) = x$, $F(f) = E_x$ commutes to direct limits; this functor is denoted by $\underline{x}$.

(ii) The identity functor Id commutes to direct limits.

(iii) It is possible to make various functors from ON (or OL) into itself by combining (i) and (ii) with Example 2.1.3(i), for instance: $\underline{1}+\mathrm{Id}$, $\mathrm{Id}+\underline{1}$, $\mathrm{Id}+\mathrm{Id}$, $\mathrm{Id}\cdot\mathrm{Id}$, $\underline{7}\cdot\mathrm{Id}$, $\mathrm{Id}\cdot\underline{9}$, $(\underline{1}+\mathrm{Id})^{\mathrm{Id}+\underline{1}}$, $(\underline{1}+\underline{9})^{\mathrm{Id}}$ (denoted by $\underline{10}^{\mathrm{Id}}$), etc...

(iv) let us write down explicitly the functor $\underline{10}^{\mathrm{Id}}$:

$$\underline{10}^{\mathrm{Id}}(x) = 10^x.$$

$$\underline{10}^{\mathrm{Id}}(f)(10^{u_1}\cdot n_1+\cdots+10^{u_p}\cdot n_p) = 10^{f(u_1)}\cdot n_1+\cdots+10^{f(u_p)}\cdot n_p.$$

For instance, if $f \in I(3, 7)$ is defined by $f(0) = 1$, $f(1) = 2$, $f(2) = 5$, then $\underline{10}^{\mathrm{Id}}(f)(944) = 900440$, $\underline{10}^{\mathrm{Id}}(f)(459) = 400590$, etc... (So the digit no. 0 is moved into the digit no. $1 = f(0)$, the digit no. 1 is moved into the digit no. $2 = f(1)$, the digit no. 2 is moved into the digit no. $5 = f(2)$, and we intercalate sufficiently many '0' between these digits.)

Remark 2.1.12. The functors sum, product, exponential commute to direct limits because they are finitary: if one looks to their definition, they are computable from finitary informations; on the other hand, the definition of $\hat{f}$ requires an infiite operation (a supremum). So commutation to direct limits is a way of saying that the functor is defined in a finitary way.

Definition 2.1.13. Let x be an ordinal; the *canonical system of* x is the direct system of Theorem 1.3.9.

Theorem 2.1.14. *Let F be a functor from* ON$<\omega$ *into* ON; *then F can be extended into a functor from* ON$\leq x$ *into* ON *commuting to direct limits iff* $\varinjlim(F(x_i), F(f_{ij}))$ *exists in* ON, *where* (x_i, f_{ij}) *is the canonical system of* x.

Proof. Let G be an extension of F into a functor from OL to itself by direct limits, then F can be extended into a functor from ON$\leq x$ into ON iff $G(y)$ is a well-order for all $y \leq x$; now suppose that the limit of $(F(x_i), F(f_{ij}))$ exists in ON: so $G(x)$ is a well-order; if $y < x$, then $G(E_{yx}) \in I(G(y), G(x))$, hence $G(y)$ must be a well-order. The other direction of the theorem is obvious.

Theorem 2.1.15. *Let F be a functor from* ON$<\omega$ *to* ON; *then F can be extended into a functor from* ON *to* ON *commuting to direct limits iff F can be extended to a functor from* ON$<\aleph_1$ *to* ON.

Proof. One sense is trivial, so suppose that G is a functor from ON$<\aleph_1$ to ON which extends F, and let H be the extension of F by directs limits: H is a functor from OL into itself, and we show that $H(x)$ is a well-order for any ordinal x: let f be a strictly decreasing sequence in $H(x)$; choose integers p_n and $f_n \in I(p_n, x)$ such that $f(n) \in \mathrm{rg}(H(f_n))$, and let X be the union of the ranges of the functions f_n; X is denumerable; so there exists $y < \aleph_1$ and $g \in I(y, x)$ such that $X = \mathrm{rg}(g)$; since one can write $f_n = gh_n$, one may define a strictly decreasing sequence h in $H(y)$ by $H(g)(h(n)) = f(n)$; we show that $H(y)$ is a well-order, and this will conclude the proof. If (y_i, g_{ij}) is the canonical system of y, then $(G(y), G(g_i))$ enjoys Definition 1.3.3(i)–(iii) w.r.t. $(G(y_i), G(g_{ij}))$ (if $(y, g_i) = \varinjlim(y_i, g_{ij})$); but, since $(H(y), H(g_i)) = \varinjlim(H(y_i), H(g_{ij}))$, it follows from Definition 1.3.3(iv) that there exists a morphism $k \in I(H(y), G(y))$ such that ... But $G(y)$ is a well-order, so $H(y)$ is a well-order.

2.2. Commutation to pull-backs

Definition 2.2.1. The functor F from $\mathscr{C}$ to $\mathscr{D}$ *commutes to pull-backs* iff for all $\mathscr{C}$-morphism such that $f \,\&\, g$ exists, then $F(f) \,\&\, F(g)$ exists and $F(f \,\&\, g) = F(f) \,\&\, F(g)$.

Remark 2.2.2. If F is any functor from $\mathscr{C}$ to $\mathscr{D}$, and if $h = f \& g$, then $F(f)$, $F(g)$, $F(h)$ enjoy condition 1.5.1(i) of pull-backs.

Remark 2.2.3. Suppose that F is a functor from OL (or ON) into itself; for all $x \in \mathrm{OL}$ and $X \subset x$, define $F(x, X)$ by the condition $F(x, \mathrm{rg}(f)) = \mathrm{rg}(F(f))$; so $F(x, X)$ is a subset of $F(x)$; then commutation to & can be written

$$F(x, X) \cap F(x, Y) = F(x, X \cap Y) \quad \text{for all } x \text{ and } X, Y \subset x.$$

Proof. Use the representation Remark 1.5.7 of morphisms.

Examples 2.2.4. (i) The functors sum, product, exponential from ON^2 to ON commute to pull-backs.

(ii) The functor *Opp* commutes to pull-backs.

(iii) The functor $\hat{}$ does not commute to pull-backs.

Proof. (i) An analogue of Remark 2.2.3 in the case of a functor of two variables is

$$F(x, X; y, Y) \cap F(x, X'; y, Y') = F(x, X \cap X'; y, Y \cap Y')$$

(one defines $F(x, X; y, Y) \subset F(x, y)$, when $X \subset x$, $Y \subset y$, by $F(x, \mathrm{rg}(f); y, \mathrm{rg}(g)) = \mathrm{rg}(F(f, g))$); now, let us take the case of the exponential: let z be $(1+x)^{u_1} \cdot (1+t_1) + \cdots + (1+x)^{u_n} \cdot (1+t_n)$, with $y > u_1 > \cdots > u_n$, and $t_1, \ldots, t_n < x$; obviously $z \in F(x, X; y, Y)$ iff:

–all the coefficients t_i are in X;

–all the coefficients u_i are in Y.

From this, it follows at once that exp. commutes to pull-backs.

(ii) This is completely evident.

(iii) Let $f, g \in I(\omega, \omega)$ be defined by $f(n) = 2n$, $g(n) = 2n+1$, then $f \& g = E_{0\omega}$, so $\widehat{(f \& g)} = E_{1\omega+1}$; on the other hand $\hat{f} \& \hat{g} = E_{1\omega} + E_1$, so $\hat{}$ does not commute to pull-backs.

Theorem 2.2.5. *Let F be a functor from* ON (*or* OL) *into* $\mathscr{C}$, *and assume that F commutes to direct limits, then, in order that F commutes to pull-backs, it is sufficient that the restriction of F to* $\mathrm{ON} < \omega$ *commutes to pull-backs.*

Proof. Suppose that $f_i \in I(x_i, y)$ $(i = 1, 2, 3)$ are such that $f_3 = f_1 \& f_2$; let (y^l, g^{lm}) be the canonical system of y (Definition 2.1.13), so $(y, g^l) = \varinjlim(y^l, g^{lm})$; let $k_i^l = f_i \& g^l$, $k_i^l \in I(x_i^l, y)$ where x_i^l is an integer. Define $f_i^l \in I(x_i^l, y)$ and $h_i^l \in I(x_i^l, x_i)$ by $f_i h_i^l = g^l f_i^l = f_i \& g^l = k_i^l$. If $l \subset m$, observe that $(x_i^l, k_i^l, h_i^l, g^{lm} f_i^l)$ enjoys Definition 1.5.1 w.r.t. f_i and g^m. So, by condition 1.5.1(ii), define h_i^{lm} by $k_i^l = k_i^m h_i^{lm}$, so $h_i^{lm} \in I(x_i^l, x_i^m)$ (see Fig. 12). We verify that $(x_i, h_i^l) = \varinjlim(x_i^l, h_i^{lm})$: let $k \in I(n, x_i)$; since $f_i k \in I(n, y)$, one can find m and $k' \in I(n, y^m)$ such that $f_i k = g^m k'$. Using again property 1.5.1(ii) of $f_i \& g^m$, one finds $k'' \in I(n, x_i^m)$, $k = h_i^m j''$ (see Fig. 13).

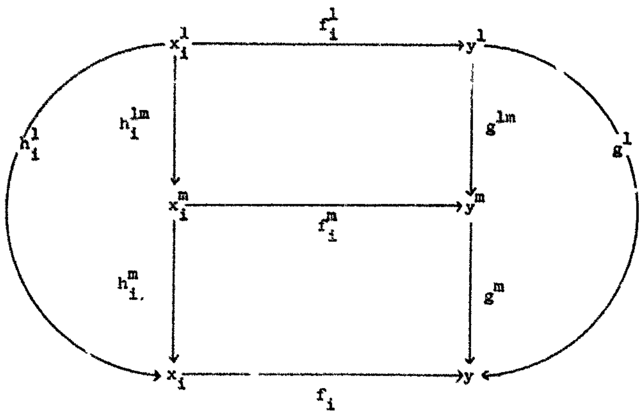


Fig. 12.

From this $x_i = \bigcup_l \mathrm{rg}(h_i^l)$. From this one get $f_i = \varinjlim(f_i^l)$. Now observe that $f_3^l = f_1^l \,\&\, f_2^l$. This is clear from the fact that $\mathrm{rg}(f_i^l) = g_l^{-1}(\mathrm{rg}(f_i))$ (obvious from the property $g^l f_i^l = f_i \,\&\, g^l$). Now, we can apply Theorem 1.5.6, and we get (using $F(f_3^l) = F(f_1^l) \,\&\, F(f_2^l)$)

$$F(f_3) = \varinjlim(F(f_3^l)) = \varinjlim(F(f_1^l)) \,\&\, \varinjlim(F(f_2^l)) = F(f_1) \,\&\, F(f_2).$$

2.3. Dilators

Definition 2.3.1. A *dilator* is a functor F from ON into itself commuting to direct limits and to pull-backs.

Remark 2.3.2. If F is a dilator, then F can be extended into a functor from OL into itself commuting to direct limits and pull-backs; we shall denote this

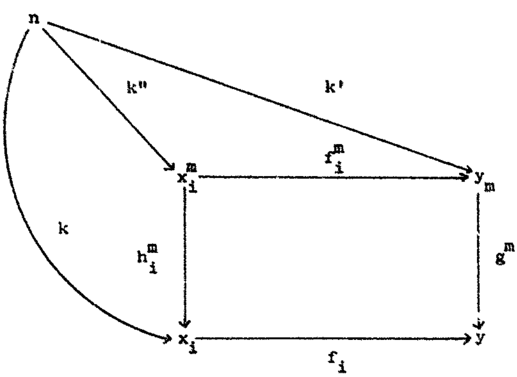


Fig. 13.

extension by the same symbol F; this extension is not unique, but this does not matter.

Examples 2.3.3. All the functors of Example 2.1.11 are dilators.

Definition 2.3.4. (i) A dilator F is *weakly finite* iff $F(n)$ is finite for all n.

(ii) Given a standard gödelnumbering of finite sequences of integers, then it is possible to encode the morphism $f \in I(n, m)$ by $\lceil f \rceil = \langle f(0), \ldots, f(n-1), m \rangle$.

(iii) A wakly finite dilator F is said to be *recursive* (resp. *primitive recursive*) iff there is a recursive (resp. primitive recursive) function φ such that, for all n, m and $f \in I(n, m)$:

$$\varphi(\lceil f \rceil) = \lceil F(f) \rceil .$$

Remarks 2.3.5. Dilators induce functions from ordinals to ordinals; if the dilator is recursive, then we have a new generalization of recursion to ordinals. By methods which will be developed in part II, we prove (see also [8, 17]) that:

(i) For a big class of admissibles, we have: if f is any function from α^+ to α^+ such that F is Σ^1 on L_{α^+}, then it is possible to find a primitive recursive dilator F such that $\forall z\ (z \in [\alpha, \alpha^+[\rightarrow \mathrm{rk}(f(z)) \leq G(z))$; this result permits to reduce α^+-recursion to computation of dilators by means of $\varinjlim$ (and to set-theoretic primitive recursion).

(ii) One can also show that $\infty 0$-recursive functions are bounded, for all values, by a recursive dilator; so, again, we have a reduction of this kind of recursion to computation of dilators by means of $\varinjlim$, and to set theoretic primitive recursion.

Remark 2.3.6. A very simple question: is it possible to majorize any function from $\aleph_1$ to $\aleph_1$ by a dilator F such that $F(\omega) < \aleph_1$? With the continuum hypothesis, this is obviously false: F is completely determined by the functions $F(f)$, for $f \in I(n, m)$, so F can be encoded by a denumerable sequence of functions from a denumerable ordinal into another denumerable ordinal; so F can be encoded by a function from N to N. If $(F_x)_{x<\aleph_1}$ is an enumeration of all these dilators, then the diagonal function $F_x(x)+1$ is not bounded by any dilator of the family.

If one assumes $V = L$, then the function $\alpha \rightsquigarrow \alpha^+$ is not bounded by any dilator F, with $F(\omega) < \aleph_1$: because, if $F' \in L_\beta$ (with $F' = F \upharpoonright \mathrm{ON} < \omega$), then clearly $F(\beta) < \beta^+$. It would be interesting to know what happens under odd hypotheses, e.g., AD.

Remark 2.3.7. The famous functor $0^{\#}$ is of course a dilator.

Definition 2.3.8. We define the following order relation on $I(x, y)$:

$$f \leq g \quad \text{iff } \forall z \in x\ (f(z) \leq g(z)).$$

Proposition 2.3.9. *If x, y are ordinals, if f, $g \in I(x, y)$, then $f \leqslant g$ iff there exist an ordinal z and functions $h \in I(y, z)$, $k \in I(z, z)$ such that $hg = khf$.*

Proof. (i) If $hg = khf$, then for all $t < x$, $h(g(t)) = k(h(f(t)))$; but, since z is an ordinal, $k(u) \geqslant u$ for all u, so $h(g(t)) \geqslant h(f(t))$, and from this, $g(t) \geqslant f(t)$.

(ii) Conversely, let $z = \omega^{2y}$, $h(t) = \omega^{2t+1}$; if $u < z$, write $u = u' + r$, with $u' = \sup\{\omega^{2f(t)+1};\ \omega^{2f(t)+1} \leqslant u\}$; let $v' = \sup\{\omega^{2g(t)+1};\ \omega^{2f(t)+1} \leqslant u\}$, and let $k(u) = v' + r$; obviously $hg = khf$, so it remains to show that k is strictly increasing: assume that, with obvious notations, $u_1 + r_1 < u_2 + r_2$;

(a) if $u_1 = u_2$, then $v_1 = v_2$, so $v_1 + r_1 < v_2 + r_2$.

(b) if $u_1 < u_2$, then $v_1 < v_2$; choose t such that $r_1 < \omega^{2f(t)+1} \leqslant u_2$; then $r_1 < \omega^{2f(t)+1} \leqslant \omega^{2g(t)+1} \leqslant v_2$, so $v_1 + r_1 < v_2 \leqslant v_2 + r_2$.

Proposition 2.3.10. *Suppose that F is a dilator, then $f \leqslant g \to F(f) \leqslant F(g)$.*

Proof. If $f \leqslant g$, then $hg = khf$, so $F(h)F(g) = F(k)F(h)F(f)$, hence $F(f) \leqslant F(g)$.

Remark 2.3.11. The result of Proposition 2.3.10 still holds when f and g are morphisms in OL: if f, $g \in I(x, y)$, it is possible to find direct systems of integers

$$(x, h_i) = \varinjlim_I (x_i, h_{ij}), \qquad (y, k_i) = \varinjlim_I (y_i, k_{ij}) \quad \text{and} \quad f_i,\ g_i \in I(x_i, y_i)$$

such that $f = \varinjlim_I (f_i)$, $g = \varinjlim_I (g_i)$; then $f \leqslant g$ iff $f_i \leqslant g_i$ for all $i \in I$.

$$f \leqslant g \to \forall i\ (f_i \leqslant g_i) \to \forall i\ (F(f_i) \leqslant F(g_i)) \to (F(f) \leqslant F(g)).$$

But the result fails for functors from OL to OL which are not extensions of dilators, for instance, $f \leqslant g \to Opp(f) \geqslant Opp(g)$.

Theorem 2.3.12 (Normal form theorem). *Let F be a dilator, and let x be an ordinal, then F induces a system of denotation of all ordinals $< F(x)$ by the pair of a finite sequence of ordinals $< x$, and of an ordinal $< F(n)$, where n is the length of the finite sequence. (So, if F is weakly finite, this component of the pair is an integer.)*

Proof. Since F commutes to direct limits, and $z < F(x)$, there exists n and $f \in I(n, x)$, such that $z \in \mathrm{rg}(F(f))$; assume that n is minimum with this property; then we claim that f is uniquely determined: if $z \in \mathrm{rg}(F(g))$, then $z \in \mathrm{rg}(F(f)) \,\&\, F(g)) = \mathrm{rg}(F(f \,\&\, g))$, since F commutes to pull-backs; if $f \neq g$, then $f \,\&\, g \in I(m, x)$, with $m < n$, contradiction with the choice of n.

It is therefore possible to represent z by $(z_0; f(0), \ldots, f(n-1))$, with z_0 defined by $F(f)(z_0) = z$. If F is weakly finite, z_0 is finite.

Observe that commutation to $\varinjlim$ corresponds to the existence of the denotation, and commutation to & to the unicity.

Remarks 2.3.13. (i) Theorem 2.3.12 still holds for OL.

(ii) It is more natural to keep in mind the ordinal x such that $z < F(x)$, so we shall use the following denotation:

$$z = (z; f(0), \dots, f(n-1); x)_F.$$

(Usually the subscript F is omitted.)

(iii) The functor F acts as follows on denotations: if $f \in I(x, y)$, then $F(f)((z_0; x_0, \dots, x_{n-1}; x)) = (z_0; f(x_0), \dots, f(x_{n-1}); y)$. (Indeed, if $h \in I(x, y)$, is defined by $h(0) = x_0, \dots, h(n-1) = x_{n-1}$, then $(z_0; x_0, \dots, x_{n-1}; x)$ represents $z = F(h)(z_0)$, and we want to prove that $(z_0; f(x_0), \dots, f(x_{n-1}); y)$ represents $F(f)(z) = F(fh)(z_0)$; since $fh \in I(n, y)$ we show that the smallest integer m such that $F(f)(z) \in \mathrm{rg}(F(g))$ for some $g \in I(m, y)$ is n, because this will give the representation $(z_0; f(x_0), \dots, f(x_{n-1}); y)$ for $F(f)(z)$; if m is minimum, then $\mathrm{rg}(g) \subset \mathrm{rg}(fh)$, so it is possible to write $g = fh'$, with $h' \in I(m, x)$; it is obvious that $z \in \mathrm{rg}(F(h'))$, and this entails $m = n$.)

(iv) Let us sum up under which conditions a sequence $(z_0; x_0, \dots, x_{n-1}; x)$ is a denotation:

(1) $x_0, \dots, x_{n-1}$ is arbitrary sequence such that $x_0 < \cdots < x_{n-1} < x$;

(2) $z_0 \in F(n)$;

(3) there is no $m < n$ and $f \in I(m, n)$ such that $z_0 \in \mathrm{rg}(F(f))$.

So, $(z_0; x_0, \dots, x_{n-1}; x)$ is a denotation iff (1)–(3) hold, and then, it denotes $F(h)(z_0)$, where $h \in I(n, x)$ is defined by $h(0) = x_0, \dots, h(n-1) = x_{n-1}$.

What is essential in the denotation is the pair $(z_0; n)$: if we know the set of all pairs $(z_0; n)$ enjoying (2) and (3), then we know all possible denotations coming from F. In Section 4, this set will be denoted by $\mathrm{rg}(F)$: it has properties analogous to the set of elements of an ordinal.

(v) It is an immediate consequence of (iii) and Proposition 2.3.10, that, if $(z_0; x_0, \dots, x_{n-1}; x)$ and $(z_0; y_0, \dots, y_{n-1}; x)$ are denotations, and if $x_0 \leq y_0, \dots, x_{n-1} \leq y_{n-1}$, then

$$(z_0; x_0, \dots, x_{n-1}; x) \leq (z_0; y_0, \dots, y_{n-1}; x).$$

Remarks 2.3.14. The Cantor normal form of base, say 10, permits to define the functor $\underline{10}^{\mathrm{Id}}$ (see Remark 2.1.11(iv)); conversely, the functor $\underline{10}^{\mathrm{Id}}$ induces the following representation: $10^{\omega+9} \cdot 7 + 10^{\omega} \cdot 5 + 10^{432} \cdot 1 + 10^{18} \cdot 5$ will be represented by $(7515; 18, 432, \omega, \omega+9; x)$ (x is any ordinal $> \omega+9$, x does not matter in that case, see Section 2.4); this sequence can be read as follows: take the 'configuration no. 7515'; this means all expressions that can be obtained as $F(f)(7515)$, and more concretely, this means any expression of the form

$$10^{x_3} \cdot 7 + 10^{x_2} \cdot 5 + 10^{x_1} \cdot 1 + 10^{x_0} \cdot 5, \quad \text{with } x_3 > \cdots > x_0,$$

and take the particular values $x_0 = 18$, $x_1 = 432$, $x_2 = \omega$, $x_3 = \omega + 9$ of the parameters. (Similarily, one would get the representations $400590 = (459; 1, 2, 5; x)$, 16000708900010010 = (1678911; 1, 4, 8, 9, 11, 15, 16; x)...) So, the theorem shows the very close relation between functors commuting to direct limits and to pull-backs, and normal form theorems such as the Cantor Normal Form Theorem. As far as we know, the concept of dilator 'catches' the mathematical structure of the normal form theorems.

Proposition 2.3.15. *Suppose that T is a natural transformation from F to G; then*

$$T(x)(z_0; x_0, \dots, x_{n-1}; x)_F = (T(n)(z_0); x_0, \dots, x_{n-1}; x)_G.$$

Proof. If $f \in I(n, x)$ is defined by $f(0) = x_0, \dots, f(n-1) = x_{n-1}$, then $(z_0; x_0, \dots, x_{n-1}; x)_F = F(f)(z_0)$; if we call this point z, then $T(x)(z) = T(x)F(f)(z_0) = G(f)T(n)(z_0)$. So it remains only to prove that n is the smallest integer m such that $T(x)(z) = G(g)(z')$ for some $z' \in G(m)$ and $g \in I(m, x)$: suppose that m has been chosen minimum with this property, and that $m < n$. Since we have $g \& f = g$ (because m is minimum), one can write $g = fh$ for some $h \in I(m, n)$, and it is possible to define $g_1, g_2 \in I(n, n+1)$ such that $g_1 \neq g_2$, but $g_1 h = g_2 h$; then:

$$G(g_1)T(n)(z_0) = G(g_1 h)(z') = G(g_2 h)(z') = G(g_2)T(n)(z_0),$$

but we have also

$$\begin{aligned} F(g_1)(z_0) &= (z_0; g_1(0), \dots, g_1(n-1); n+1)_F \\ &\neq (z_0; g_2(0), \dots, g_2(n-1); n+1)_F = F(g_2)(z_0), \end{aligned}$$

so $T(n+1)F(g_1)(z_0) \neq T(n+1)F(g_2)(z_0)$, and we have obtained a contradiction (since $T(n+1)F(g_i) = G(g_i)T(n)$).

So the hypothesis $m < n$ is absurd, and we have therefore proved that

$$T(x)(z_0; x_0, \dots, x_{n-1}; x)_F = (T(n)(z_0); x_0, \dots, x_{n-1}; x)_G$$

Remark 2.3.16. So, a natural transformation T is completely determined by its action on the ranges of F and G (Remark 2.3.13(iv)), i.e., T is determined by the function $t((z_0; n)) = (T(n)(z); n)$, from $\mathrm{rg}(F)$ to $\mathrm{rg}(G)$. In Section 4 we shall introduce $\mathrm{rg}(T) = \mathrm{rg}(t)$. $\mathrm{rg}(T)$ determines uniquely T, and given any $X \subset \mathrm{rg}(F)$, there is an unique T such that $\mathrm{rg}(T) = X$. $\mathrm{rg}(T)$ is analogue to the usual range $\mathrm{rg}(f)$ of a function.

Proposition 2.3.17. *Let F be a dilator, and suppose that $(z; x_0, \dots, x_{n-1}; x)$ and $(z'; y_0, \dots, y_{m-1}; x)$ are denotations for elements $< F(x)$; then the order relation between these two elements depends only on:*
– z and z', and n and m;
– the order relations between the points x_i and y_j;

Proof. The meaning of the theorem is that, given any strictly increasing sequences $x'_0 < \cdots < x'_{n-1} < x$ and $y'_0 < \cdots < y'_{m-1} < x$, such that for all i, j, $x_i \leq y_j$ iff $x'_i \leq y'_j$, then, with

$$t = (z; x_0, \ldots, x_{n-1}; x), \qquad u = (z'; y_0, \ldots, y_{m-1}; x),$$
$$t' = (z; x'_0, \ldots, x'_{n-1}; x) \quad \text{and} \quad u' = (z'; y'_0, \ldots, y'_{m-1}; x),$$

$t < u$ iff $t' < u'$. Let $z_0, \ldots, z_{k-1}$ (resp. $z'_0, \ldots, z'_{k-1}$) be the enumeration of the points $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}$ (resp. $x'_0, \ldots, x'_{n-1}, y'_0, \ldots, y'_{m-1}$) in increasing order, let $y = x \cdot (k+1)$, and define $f, f' \in I(x, y)$ by

$$f(a) = a \quad \text{if } a < z_0,$$
$$f(z_i + a) = x \cdot (i+1) + a, \quad \text{if } z_i + a < z_{i+1} \quad \text{or} \quad i = k-1,$$
$$f'(a) = a \quad \text{if } a < z'_0,$$
$$f'(z'_i + a) = x \cdot (i+1) + a, \quad \text{if } z'_i + a < z'_{i+1} \quad \text{or} \quad i = k-1.$$

Then $F(f)(t) = F(f')(t')$, and $F(f)(u) = F(f')(u')$, so $t < u$ iff $t' < u'$.

Definition 2.3.18. The following data define a category PN:
–objects: ordinals,
–morphisms from x to y: $f \in P(x, y)$ iff f is a strictly increasing function from a subset of x to y.

If $f \in P(x, y)$, one defines $f^{-1} \in P(y, x)$ by the condition: $f^{-1}(u)$ is defined iff $u \in \mathrm{rg}(f)$ and $f^{-1}(u) = t$ iff $f(t) = u$.

Theorem 2.3.19. (i) *If F is a dilator, then there exists an unique extension of F to* PN. *F satisfies $F(f^{-1}) = F(f)^{-1}$.*

(ii) *If T is a natural transformation between the dilators F and G, then T is still a natural transformation between their extensions to* PN.

Proof. (i) f^{-1} is uniquely determined by $gfg = g$ and $fgf = f$, so $F(f^{-1}) = F(f)^{-1}$ if F can be defined on partial functions; if $f \in P(x, y)$, then f can be written uniquely $f = gh^{-1}$, with g, h total, so $F(f) = F(g)F(h)^{-1}$ is uniquely determined. We need to show the existence: assume that $f \in P(x, y)$, and that $z < F(x)$, so $z = (z_0; x_0, \ldots, x_{n-1}; x)$; if $f(x_i)$ is undefined for some $i < n$, then $F(f)(z)$ is undefined; otherwise, we put $F(f)(z) = (z_0; f(x_0), \ldots, f(x_{n-1}); y)$. If $z < z'$, and $F(f)(z)$ and $F(f)(z')$ are both defined, then $F(f)(z) < F(f)(z')$: this comes from Proposition 2.3.17.

(ii) This is an immediate consequence of Proposition 2.3.15: if $f \in P(x, y)$, then $T(y)F(f) = G(f)T(x)$.

2.4. *Flowers*

In this section, we shall make some connections between the topological notion of continuity, and the categorical notion (i.e., commutation to direct limits).

Definition 2.4.1. A *flower* is a dilator which enjoys the property:

(FL) for all $x, y \in \mathrm{ON}$, with $x \leq y$, $F(E_{xy}) = E_{F(x)F(y)}$.

Proposition 2.4.2. *The dilator F is a flower iff F enjoys the property: for all n, m, with $n \leq m$, $F(E_{nm}) = E_{F(n)F(m)}$.*

Proof. Suppose that $x \leq y$ and let $(y_l, g_{lm})_{l,m \in L}$ be the canonical system of y, so $(y, g_l) = \varinjlim(y_l, g_{lm})$; define $x_l = \mathrm{card}(x \cap l)$, $f_{lm}(p) = q$ iff the pth element of $x \cap l$ is the qth element of $x \cap m$, $f_l(p) = p$th element of $x \cap l$; so $(x, f_l) = \varinjlim(x_l, f_{lm})$, and, obviously, $E_{xy} = \varinjlim(E_{x_ly_l})$. So

$$F(E_{xy}) = \varinjlim F(E_{x_ly_l}) = \varinjlim E_{F(x_l)F(y_l)};$$

in order to show that the last expression is equal to $E_{F(x)F(y)}$, it suffices to remark that a direct limit $\varinjlim(h_l)$ of morphisms of the kind $E_{t_lu_l}$ is again a morphism of the kind E_{tu}.

Theorem 2.4.3. *Suppose that F is a flower, then then the function $x \rightsquigarrow F(x)$ from ordinals to ordinals is topologically continuous.*

Proof. Topological continuity means that, if x is limit, then $F(x) = \sup\{F(y); y < x\}$; now recall (Example 1.3.6(ii)) that $(x, E_{yx}) = \varinjlim(y, E_{yy'})_{y,y'<x}$. From this one gets:

$$(F(x), F(E_{yx})) = \varinjlim(F(y), F(E_{yy'})) = \varinjlim(F(y), E_{F(y)F(y')}),$$

but this forces $F(x) = \sup\{F(y); y < x\}$.

Remark 2.4.4. Theorem 2.4.3 gives the precise relation between continuity and commutation to direct limits; in general the two concepts are not comparable. However, one must say that the concept of being continuous topologically is more superficial than the other concept, simply because topological continuity is just a property of the limit points, whereas commutation to direct limits implies that the function is completely determined by a fixed denumerable set of data (i.e., the restriction of F to $\mathrm{ON}^{<\omega}$). A remark similar to Remark 2.3.7 would show the existence of continuous functions from $\aleph_1$ to $\aleph_1$ which are not bounded by any flower F such that $F(\omega) < \aleph_1$.

Examples 2.4.5. (i) The functors Id, $\underline{x}$ are flowers.

(ii) Let F be one of the binary functors sum, product, exponential, then;
–if x is an ordinal, then the functor F_x defined by $F_x(y) = F(x, y)$, $F_x(g) = F(E_x, g)$, is a flower;
–is y is an ordinal, then the functor F^y defined by $F^y(x) = F(x, y)$, $F^y(f) = F(f, E_y)$ is in general not a flower.

(iii) From (i) and (ii), it is easy to build flowers by composition. For instance $\underline{10}^{\mathrm{Id}}$ is a flower.

Proof. (i) and (iii) are obvious. The functions $x \rightsquigarrow x+y$, $x \rightsquigarrow x \cdot y$, $x \rightsquigarrow (1+x)^y$ are not continuous (except for exceptional values of y), so F^y cannot enjoy (FL). The fact that $F(E_x, E_{yy'}) = E_{F(x,y)F(x,y')}$ is immediate from the definition:

$$\begin{aligned}&(1+E_x)^{E_{yy'}}((1+x)^{u_1}\cdot(1+t_1)+\cdots+(1+x)^{u_p}\cdot(1+t_p))\\&\quad=(1+x)^{u_1}\cdot(1+t_1)+\cdots+(1+x)^{u_p}\cdot(1+t_p)).\end{aligned}$$

Remark 2.4.6. In Definition 1.1.4, the functors are flowers 'in y' but not 'in x'. This can be seen directly on the definition, which depends explicitly on x and x' but not on y or y'. This is made precise by the following result on notations.

Proposition 2.4.7. *F being a dilator, then F is a flower iff* $(z; x_0, \ldots, x_{n-1}; x) = (z; x_0, \ldots, x_{n-1}; y)$ *for all denotation* $(z; x_0, \ldots, x_{n-1}; x)$ *and all* $y > x$.

Proof. This is immediate from Remark 2.3.13(iii) since

$$F(E_{xy})((z; x_0, \ldots, x_{n-1}; x)) = (z; x_0, \ldots, x_{n-1}; y).$$

Remark 2.4.8. It follows from Proposition 2.4.7 that if F is a flower, then the last component x in $(z; x_0, \ldots, x_{n-1}; x)$ is redundant. So, in the case of a flower (for instance $\underline{10}^{\mathrm{Id}}$), objects will be perfectly denoted by $(z; x_0, \ldots, x_{n-1})$, and we shall have therefore the following representation of the action of $F(f)$:

$$F(f)(z; x_0, \ldots, x_{n-1}) = (z; f(x_0), \ldots, f(x_{n-1})).$$

Examples 2.4.9. (i) If F is a dilator, then define $G = \int F$ by $G(x) = \sum_{y<x} F(y)$, and, if $f \in I(x, x')$, $z < F(y')$, $y' < x$:

$$G(f)\Big(\sum_{y<y'} F(y) + z\Big) = \sum_{y<f(y')} F(y) + F(g)(z), \quad \text{where } g \in I(y', f(y'))$$

is defined by $g(u) = f(u)$ for all $u < y'$. Condition (FL) is obviously satisfied; the point $\sum_{y<y'} F(y) + z$ (with $z \in F(y')$) can be written $G(f+E_1)(\sum_{y<n} F(y) + z_0)$, when $z = F(f)(z_0)$ and $f \in I(n, y')$ (such n, f exist, since F commutes to direct limits). So by Corollary 2.1.8(ii), G commutes to $\varinjlim$. Finally, we can express the set $G(x, X)$ (that we shall denote $G(X)$, since it is a consequence of (FL) that these sets are independant of the choice of x such that $X \subset x$). The point $\sum_{y<y'} F(y) + z$ is in $G(X)$ iff (with $z < F(y')$) $y' \in X$ and $z \in F(y', X \cap y')$, and from this, commutation to pull-backs is immediate (i.e., $G(X) \cap G(X') = G(X \cap X')$).

Conversely, if G is a flower, then one can find an unique ordinal a and an unique dilator $F = \partial G$ such that $G = \underline{a} + \int F$, i.e., for all x, $G(x) = a + (\int F)(x)$, for all $f \in I(x, y)$, $G(f) = E_a + (\int F)(f)$. Obviously, one must take $a = G(0)$ (because $(\int F)(0) = 0$). Define F by $G(x) + F(x) = G(x+1)$; $G(f) + F(f) = G(f + E_1)$: this is possible, since, if $f \in I(x, y)$, $G(f+E_1)$ extends $G(f)$: if $z \in G(x)$, then $z = G(E_{xx+1})(z)$ by (FL), so

$$G(f+E_1)(z) = G(f+E_1)G(E_{xx+1})(z) = G(E_{yy+1})G(f)(z) = G(f)(z)$$

(we have used $(f+E_1)E_{xx+1} = E_{yy+1}f$, and (FL) applied to E_{yy+1}).

One verifies easily that F is a functor from ON to ON; and that F commutes to $\varinjlim$ and &. So, F is a dilator, and clearly

$$G = \underline{a} + \int F.$$

Just a word concerning the relations between the denotations for F and for $G = a + \int F$: if

$$z = a + \sum_{<y'} F(y) + z', \quad \text{with } z' < F(y'),$$

then $z = (z_0; x_0, \ldots, x_{n-1}, y')_G$ for some sequence $x_0 < \cdots < x_{n-1} < y'$, whereas $z' = (z_0'; x_0, \ldots, x_{n-1}; y')_F$ for the same sequence $x_0 < \cdots < x_{n-1} < y'$: when G is built from F, then ';' is replaced by ','.

(ii) Define a functor V from $(\mathrm{ON} < \omega) \times \mathrm{ON}$ to ON by

$$V(0, x) = x, \qquad V(1, x) = \omega^x,$$
$$V(n+p, x) = V(n, V(p, x)) \quad \text{with } n < \omega, p < \omega, x \in \mathrm{ON}.$$

If $f \in I(n, m)$, $f' \in I(p, q)$, $g \in I(x, y)$:

$$V(E_0, g) = g, \qquad V(E_{01}, g)(z) = \omega^{g(z)}; \qquad V(E_1, g) = \underline{\omega}^g (= (1+E_\omega)^g),$$
$$V(f+f', g) = V(f, V(f', g)).$$

Then this functor can be extended into a functor from ON×ON to ON commuting to $\varinjlim$ and &; furthermore, the functor has the following properties:

(1) the functors $V(a, \cdot)$ are flowers;

(2) the functor $V(\cdot, 0)$ is a flower;

(3) $V(\omega^x, y) = K^x(y)$, where K is the Veblen hierarchy.

The reader will find a detailed exposition in [9]. See also Section 5.3.

(iii) Let F be a dilator, and define a functor G from ON×ON to ON:

$$G(x, 0) = x, \qquad G(f, E_0) = f, \tag{1}$$

$$G(x, y+1) = G(x, y) + F(G(x, y)), \qquad G(f, g+E_1) = G(f, g) + F(G(f, g)), \tag{2}$$

$$G(f, g+E_{01}) = G(f, g) + E_{\mathrm{OF}(G(x', y'))}, \tag{3}$$

with $f \in I(x, x')$, $g \in I(y, y')$:

$$G(x, \sup(y_i)) = \sup(G(x, y_i)), \qquad G\Big(f, \bigcup_i g_i\Big) = \bigcup_i G(f, g_i) \tag{4}$$

(the union of a family of morphisms is a special case of direct limit of a system of morphisms; see Example 1.3.8(i)).

G commutes to $\varinjlim$: suppose that $z \in G(x, y)$. We prove that $z \in \mathrm{rg}(G(f, g))$ for some $f \in I(n, x)$, $g \in I(m, y)$, by induction on y:
–if $y = 0$, let $n = 1$, $f(0) = z$, $g = E_0$;
–if $y = y' + 1$, and $z \in G(x, y')$, then by induction hypothesis $z \in \mathrm{rg}(G(f, g))$, with $g \in I(m, y')$, so $z \in \mathrm{rg}(G(f, g + E_1))$. If $z = G(x, y') + z'$, then $z' \in \mathrm{rg}(F(h))$ for some $h \in I(p, G(x, y'))$. The induction hypothesis applied to the elements of $\mathrm{rg}(h)$ gives functions f and g, $f \in I(n, x)$, $g \in I(m, y')$ such that $\mathrm{rg}(h) \subset \mathrm{rg}(G(f, g))$, and so $z \in \mathrm{rg}(G(f, g + E_1))$.
–if y is limit, then $z < G(x, y')$ for some $y' < y$; then by induction hypothesis, there is $f \in I(n, x)$, $g \in I(m, y')$ such that $z \in \mathrm{rg}(G(f, g))$; then $z \in \mathrm{rg}(G(f, g + E_{0y-y'}))$.

G commutes to &: we express the sets $G(x, X, Y)$, where X and Y are sets of ordinals and $X \subset x$:

$$G(x, X, \emptyset) = X; \qquad G\Big(x, X, \bigcup_i Y_i\Big) = \bigcup_i G(x, X, Y_i)$$

$$G(x, X, Y \cup \{a\}) = G(x, X, Y) \cup \{G(x, a) + z;\ z \in F(G(x, a), G(x, X, Y))\}.$$

It is immediate that $G(x, X, Y) \cap G(x, X', Y') = G(x, X \cap X', Y \cap Y')$.

G enjoys (FL) in the variable y: this is immediate (and already used implicitely in the notation $G(x, X, Y)$ which does not mention y).

Now, if one takes $a \in \mathrm{ON}$, then the functor $G(a, x)$, $G(E_a, f)$ is a flower. (If $F \neq \underline{0}$, G is what we shall call later a *bilator*).

Remark 2.4.10. If F is a non constant flower, then we have the following properties (we use the simplified denotation $(z; x_0, \ldots, x_{n-1})$ of Remark 2.4.8):

(i) If $z = (z_0; x_0, \ldots, x_{n-1})$, then $z < F(x)$ iff $x_0 \cdots x_{n-1} < x$ (i.e., $n = 0$ or $x_{n-1} < x$ (because $z < F(x)$ iff $z \in \mathrm{rg}(F(E_x))$)).

(ii) Since F is non constant, then there is a denotation $z = (z_0; x_0, \ldots, x_{n-1})$, with $n \neq 0$. Let $a = (a_0; x_0, \ldots, x_{n-1})$ be the smallest such denotation, then $x_0 = 0, \ldots, x_{n-1} = n - 1$ (apply Proposition 2.3.10), and, by (i) above; $a = F(0) = F(1) = \cdots = F(n-1) \neq F(n)$. If $x \geqslant n - 1$, then $F(x) < F(x + 1)$ (because

$$(a_0; 0, \ldots, n-2, x) \in F(x+1) - F(x)).$$

(iii) Another consequence of (i) is that if $f \in I(x, y)$, $z \leqslant x$, and $f(z') = z'$ for all $z' < z$, then $F(f)(t) = t$ for all $t < F(z)$.

(iv) Yet another consequence of (i) is that $F(x)$ is the smallest $z = (z_0; x_0, \ldots, x_{p-1})$ such that $x_{p-1} \geqslant x$. If $x \geqslant n - 1$ (n is the integer defined in (ii)), then $x_{p-1} = x$ suffices. It is immediate that $F(x) = (z_0; 0, \ldots, p-2, x_{p-1})$, with $x_{p-1} = \sup(x, p-1)$.

(v) If $F(\omega) = (b_0; 0, \ldots, m-2, \omega)$, then, for all $x \geqslant m - 1$, $F(x) = (b_0; 0, \ldots, m-2, x)$ (because if $F(x) = (c; 0, \ldots, p-2, x)$, then by (iv)

$$F(\omega) = (b_0; 0, \ldots, m-2, \omega) \leqslant (c; 0, \ldots, p-2, \omega),$$

so, by Proposition 2.3.17,

$$(b_0; 0, \ldots, m-2, x) \leq (c; 0, \ldots, p-2, x) = F(x),$$

so by (iv) again $F(x) = (b_0; 0, \ldots, m-2, x)$.

Theorem 2.4.11. (i) *If F is a non constant flower, there exists an integer k and a natural transformation T from* Id *to* $F \circ (\underline{k} + \mathrm{Id})$.

(ii) *Moreover, k may be chosen in such a way that $T(x)(z) = F(k+z)$ for all $x \in \mathrm{ON}$ and $z < x$.*

Proof. (i) Define $a = (a_0; 0, \ldots, n-1)$ as in Remark 2.4.10(ii), and $k = n-1$, $T(x)(z) = (a_0; 0, \ldots, n-2, k+z)$.

(ii) Let $b = (b_0; 0, \ldots, m-1)$ as in Remark 2.4.10(v) and $k = m-1$, $T(x)(z) = (b_0; 0, \ldots, m-2, k+z)$, then $T(x)(z) = F(k+z)$.

Remark 2.4.12. Theorem 2.4.11(ii) can be restated as follows: the flower $G = F \circ (\underline{k} + \mathrm{Id})$ enjoys the property: For all x, z with $z < x$, then $G(z) < G(x)$, and if $f \in I(x, y)$, then

$$G(f)(G(z)) = G(f(z)).$$

3. The principle of induction on dilators

3.1. Decomposition of a dilator

Definition 3.1.1. (i) Let x be an ordinal, and let $(F_z)_{z<x}$ be a family of dilators, then it is possible to define a new dilator $G = \sum_{z<x} F_z$ by $G(a) = \sum_{z<x} F_z(a)$, and, if $f \in I(a, b)$, if $z' < x$ and $u < F_{z'}(a)$:

$$G(f)\Big(\sum_{z<z'} F_z(a) + u\Big) = \sum_{z<z'} F_z(b) + F_{z'}(f)(u).$$

(ii) Let y be another ordinal, let $(G_z)_{z<y}$ be another family of dilators, and let (T_z) be a family of natural transformations from F_z to $G_{f(z)}$, where f is a function in $I(x, y)$, then one defines a natural transformation $T = \sum_{z<f} T_z$ from $\sum_{z<x} F_z$ to $\sum_{z<y} G_z$ by

$$T(a)\Big(\sum_{z<z'} F_z(a) + u\Big) = \sum_{z<f(z')} G_z(a) + T_{z'}(a)(u) \quad (z' < x, u < F_{z'}(a)).$$

Proposition 3.1.2. Definition 3.1.1(i) *and* (ii) *are correct definitions.*

Proof. The fact that (i) and (ii) define functors and natural transformations is more or less obvious. We show that the functor defined by (i) commutes to $\varinjlim$: if

$t \in G(a)$, write $t = \sum_{z<z'} F_z(a) + u$; if $n, v < F_{z'}(n)$, $f \in I(n, a)$ are such that $u = F_{z'}(f)(v)$, then, with $t' = \sum_{z<z'} F_z(n) + v$, one has clearly $t = G(f)(t')$. To prove commutation to & amounts essentially to express the sets $G(x, X)$ when $X \subset x$; but, it is obvious that $\sum_{z<z'} F_z(a) + u \in G(x, X)$ iff $u \in F_{z'}(x, X)$. From this it is immediate that $G(x, X) \cap G(x, Y) = G(x, X \cap Y)$.

Remark 3.1.3. If $x = 2$, then we obtain the definition of the sum of two dilators $F + F'$. This definition can also be obtained by means of composition with the binary functor $+$, since $(F+F')(a) = F(a) + F'(a)$, $(F+F')(f) = F(f) + F'(f)$.

Definition 3.1.4. A dilator is *perfect* iff it is non zero, and if given by decomposition $F = F' + F''$, then $F' = \underline{0}$ or $F'' = \underline{0}$.

Theorem 3.1.5. (i) *Let F be a dilator, then there exists an unique ordinal i and an unique family $(F_z)_{z<i}$ of perfect dilators such that $F = \sum_{z<i} F_z$.*

(ii) *Furthermore, if $G = \sum_{z<j} G_z$ is a sum of perfect dilators and if T is a natural transformation from F to G, then there is an unique $h \in I(i, j)$ and an unique family $(T_z)_{z<i}$ of natural transformations from F_z to $G_{h(z)}$ such that $T = \sum_{z<h} T_z$.*

Proof. (i) Consider the class of all pairs (x, z) with $z < F(x)$, and define a preorder on this class by (Proposition 2.3.17. is essential to understand this definition):

$$(x, (z_0; x_0, \ldots, x_{n-1}; x)) \prec (x'; (z'_0; x'_0, \ldots, x'_{m-1}; x'))$$

$$\text{iff } (z_0; 0, \ldots, n-1; n+m) \leqslant (z'_0; n, \ldots, n+m-1; n+m)$$

(equivalently, $(z_0; x_0, \ldots, x_{n-1}; x + x') \leqslant (z'_0; x + x'_0, \ldots, x + x'_{m-1}; x + x')$).

Let $\equiv$ be the associated equivalence relation; if c is an equivalence class, and x is an ordinal, define $c_x = \{z; (x, z) \in c\}$, then

(1) if $f \in I(x, y)$, then $F(f)$ maps c_x into c_y, simply because $(x, z) \equiv (y, F(f)(z))$.

(2) the quotient $\prec/\equiv$ is a well-order; if c is an equivalence class, then $c_\omega \neq \emptyset$ (because $(x; (z_0; x_0, \ldots, x_{n-1}; x)) \equiv (\omega; (z_0; 0, \ldots, n-1; \omega))$. So it suffices to consider the restriction of $\prec$ to pairs (ω, z), and to remark that $z \leqslant z' \rightarrow (\omega, z) \prec (\omega, z')$, hence c_ω is a non-void interval.

(3) Let i be the ordinal isomorphic with $\prec/\equiv$, and c^z, for $z < i$, be the zth equivalence class modulo $\equiv$; we define: $F_z(a) =$ ordinal isomorphic to c^z_a, and, if $f \in I(a, b)$, $F_z(f)(t) = u$ iff the image under $F(f)$ of the tth element of c^z_a is the uth element of c^z_b.

(4) It is immediate that F_z is a dilator for all $z < i$ and that $F = \sum_{z<i} F_z$.

(5) Assume that F_z is not perfect for some z, then $F_z = G + G'$ with G' non zero. The points $(\omega, \sum_{z'<z} F_{z'}(\omega))$ and $(\omega, \sum_{z'<z} F_{z'}(\omega) + G(\omega))$ are in c^z, whereas it is

immediate that

$$G(\omega)\neq 0 \to \Big(\omega, \sum_{z'<z} F_{z'}(\omega)+G(\omega)\Big) \not< \Big(\omega, \sum_{z'<z} F_{z'}(\omega)\Big),$$

hence $G(\omega)=0$, i.e., $G=\underline{0}$.

(6) The unicity of the decomposition in sum of perfect dilators is left to the reader.

(ii) If T is a natural transformation, then T is compatible with the preorders $<_F$ and $<_G$ associated with F and G: if $(x,z)<_F(x',z')$, then $(x, T(x)(z))<_G (x', T(x')(z'))$; if

$$(z_0; 0,\ldots,n-1; n+m)_F \leqslant (z_0'; n,\ldots,n+m-1; n+m)_F,$$

then

$$T(n)(z_0); 0,\ldots,n-1; n+m)_G \leqslant (T(m)(z_0'); n,\ldots,n+m-1; n+m)_G,$$

using Proposition 2.3.15; this proves the property.

From this

$$(x,z)\equiv_F(x',z') \to (x, T(x)(z))\equiv_G (x', T(x')(z')),$$

so one can define $h\in I(i,j)$ by $h(z)=z'$ if the points in the zth equivalence class modulo $\equiv_F$ are sent by T into the z'th equivalence class modulo $\equiv_G$. Let us denote the equivalence classes modulo $\equiv_G$ by $d^{z'}$. If $z<i$, define T_z by $T_z(a)(t)=u$ iff the image under $T(a)$ of the tth element of c_a^z is the uth element of $d_a^{h(z)}$. T_z is a natural transformation from F_z to $G_{h(z)}$. It is immediate that $T=\sum_{z<h} T_z$, and that this decomposition is unique.

Remarks 3.1.6. (i) If x is limit, then $(x,z)<(x',z')$ iff there exists $f\in P(x',x)$ such that $F(f)(z')$ is defined and $F(f)(z')\geqslant z$. If $z=(z_0; x_0,\ldots,x_{n-1}; x)$, $z'=(z_0'; x_0',\ldots,x_{m-1}'; x')$, define $f\in P(x',x)$ by $f(x_i')=x_{n-1}+1+i$, $f(a)$ undefined on the other points; then $(x,z)<(x',z')$ iff $F(f)(z')\geqslant z$.

(ii) If $x'\leqslant x$ and $x=\omega^y$, with $y\neq 0$, then the function f in (i) can be chosen total: define $f(a)=x_{n-1}+1+a$.

(iii) If F is a flower, then there are only two possibilities:

$-F$ is constant, so $F=\underline{x}$, $i=x$, $F_z=\underline{1}$;

$-F$ is non constant, $i=i'+1$, $F_z=\underline{1}$ for $z<i'$, $F_{i'}$ is a perfect flower and $i'=(i'; 0,\ldots,n-1)=F(0)=\cdots=F(n-1)<F(n)$, see Remark 2.4.10(ii).

Definition 3.1.7. The dilator F is said to be

of kind 0 iff $F=\underline{0}$,

of kind 1 iff $F=F'+\underline{1}$ for some F',

of kind ω iff $F=\sum_{z<x} F_z$, with x limit, and $F_z\neq\underline{0}$ for all $z<x$,

of kind Ω iff $F=F'+F''$, with F'' perfect and $F''\neq\underline{1}$.

Proposition 3.1.8. *A dilator F is of one and only one of the kinds* $0, 1, \omega, \Omega$.

Proof. Write $F=\sum_{z<x} F_z$, with F_z perfect for all z, then, if $x=0$, then F is of kind 0; if x is limit, F is of kind ω; if $x=x'+1$, let $F'=\sum_{z<x'} F_z$, then $F=F'+F_z$ and F is of kind 1 or Ω according to $F_z=\underset{\sim}{1}$ or $F_z\neq\underset{\sim}{1}$.

Remarks 3.1.9. (i) Dilators appear as an extension of the concept of ordinal. If one prefers, a dilator is something like a 'well-ordered class', if one identifies the ordinals with the constant dilators $\underset{\sim}{x}$, then after having exhausted all the points $\underset{\sim}{x}$, there is Id, and after Id, all the points $\mathrm{Id}+\underset{\sim}{x}$, then $\mathrm{Id}+\mathrm{Id},\ldots$, then $\mathrm{Id}\cdot\underset{\sim}{x}$, then $\mathrm{Id}^2\ldots$ However, the similarity cannot be developed too far, for instance, there is no natural way of defining a linear well-order on dilators, as far as I know.

(ii) However, the decomposition Theorem 3.1.5 shows that dilators are generated by four operations. The first three operations are $\underset{\sim}{0}$, successor, and supremum, which are familiar ways of constructing ordinals from 'below'; the crucial point is operation four: adding a perfect dilator $\neq\underset{\sim}{1}$. Perfect dilators will be therefore our next goal of study.

(iii) Another meaning of Theorem 3.1.5 (combined with an analysis of perfect dilators) will be to make inductions, or recursions on dilators: typically, the functor $\bigwedge$ comes from this decomposition. These constructions by induction of functors will be something very close to the Bar-induction and Bar-recursion of type 2.

3.2. Prime dilators

Definition 3.2.1. The dilator F is *prime* iff $F\neq\underset{\sim}{0}$ and for some integer n, all points can be written $z=(0;x_0,\ldots,x_{n-1};x)$ in the F-representation.

Proposition 3.2.2. (i) *A prime dilator is perfect.*

(ii) *$F\neq\underset{\sim}{0}$ is prime iff for all G and T natural transformation from G to F, $G=\underset{\sim}{0}$ or $G=F$.*

Proof. (ii) From the expression of natural transformations (Proposition 2.3.15), it follows that: if F is prime and T is a natural transformation from G to F, then $G=\underset{\sim}{0}$ or $G=F$. Conversely, given $F\neq\underset{\sim}{0}$, then there is a dilator G defined by $G(x)=$ ordinal isomorphic to the points of the kind $(0;x_0,\ldots,x_{n-1};x)$, $G(f)(t)=u$ iff the tth element of the form $(0;x_0,\ldots,x_{n-1};x)$ is sent into the uth element $(0;y_0,\ldots,y_{n-1};y)$ by $F(f)$, when $f\in I(x,y)$. Then $T(x)$ defined by $T(x)((0;x_0,\ldots,x_{n-1};x)_G)=(0;x_0,\ldots,x_{n-1};x)_F$ is a natural transformation from G to F, and, if by hypothesis $G=\underset{\sim}{0}$ or $G=F$, it follows that $G=F$, so F is prime.

Definition 3.2.3. Let F be a prime dilator, and let $(0;x_0,\ldots,x_{n-1},x)$ be the associated representation, then F is said to be of *degree n*. If F is of degree n, then F induces a permutation σ_F of n:

(i) Consider the points $a_i=(0;0,2,\ldots,2i-2,\ 2i+1,\ 2i+2,\ldots,2n-2;2n)$ these points are pairwise distinct.

(ii) Define σ_F by $i<j\leftrightarrow a_{\sigma_F(i)}>a_{\sigma_F(j)}$.

Theorem 3.2.4. *If F is prime, then F is completely determined by σ_F.*

Proof. We show the more precise statement: if $(0; x_0, \ldots, x_{n-1}; x)$ and $(0; y_0, \ldots, y_{n-1}; x)$ are such that, for some $p<n$, $x_{\sigma(p)}<y_{\sigma(p)}$, and $x_{\sigma(i)}=y_{\sigma(i)}$ for all i ·, then

$$(0; x_0, \ldots, x_{n-1}; x)<(0; y_0, \ldots, y_{n-1}; x).$$

(1) We make the following extra hypothesis on $s=(0; x_0, \ldots, x_{n-1}; x)$ and $t=(0; y_0, \ldots, y_{n-1}; x)$:

$$x_{\sigma(p)}+\omega \leqslant \inf(x_{\sigma(p)+1}, y_{\sigma(p)}).$$

We argue by induction on the number k of indices i such that $y_i<x_i$:

(i) If $k=0$, then this is immediate from Proposition 2.3.10.

(ii) Suppose that $k>0$; then we shall use the following property: if $f\in P(x, y)$ is such that $F(f)(s)$ and $F(f)(t)$ are defined, then $s<t$ iff $F(f)(s)<F(f)(t)$. (Write $f=gh^{-1}$ with g, h total.) Let i be the smallest index such that $y_i<x_i$:
Then define s' by replacing x_i by y_i, and $x_{\sigma(p)}$ by $x_{\sigma(p)}+1$ in s.

The parameters in s and s' are ordered as follows:

$$x_0, \ldots, x_{i-1}, y_i, x_i, x_{i+1}, \ldots, x_{\sigma(p)}, x_{\sigma(p)}+1, x_{\sigma(p)+1}, \ldots$$

or

$$x_0, \ldots, x_{\sigma(p)}, x_{\sigma(p)}+1, \ldots, y_i, x_i, \ldots$$

Define $f\in P(2n, x)$ by $f(2j)=x_j$ except if $j=i$, $f(2i)=y_i$; $f(2i+1)=x_i$, $f(2\sigma(p)+1)=x_{\sigma(p)}+1$. f is undefined otherwise. Then $s=F(f)(a_i)$ and $s'=F(f)(a_{\sigma(p)})$. Since $\sigma^{-1}(i)>p$, $a_i<a_{\sigma(p)}$, hence $s<s'$.

Remark now that the induction hypothesis for $k-1$ gives $s'<t$, so $s<t$.

(2) In the general case, we only know that

$$x_{\sigma(p)}+1\leqslant \inf(x_{\sigma(p)+1}, y_{\sigma(p)}).$$

Consider the function $f\in I(x, \omega\cdot x)$ defined by $f=E_{1\omega}\cdot E_x$, then the images under f enjoy the condition:

$$f(x_{\sigma(p)})+\omega \leqslant \inf(f(x_{(p)+1}), f(y_{(p)})),$$

so by (1) $F(f)(s)<F(f)(t)$, hence $s<t$.

Theorem 3.2.5. *If σ is a permutation of n, then there is an unique prime dilator F such that $\sigma=\sigma_F$.*

Proof. The unicity is exactly Theorem 3.2.4, so we need only to show the existence. Consider the dilator $G=\mathrm{Id}^n$; in $G(x)=x^n$ consider all the points $x^{n-1}\cdot x_{\sigma(0)}+\cdots+x^0\cdot x_{\sigma(n-1)}$, with $x_0<\cdots<x_{n-1}<x$. We use the denotation $(\Theta; x_0, \ldots, x_{n-1}; x)$ for such a point (Θ is the point written $\sigma(0)\cdots\sigma(n-1)$ in

numeration base n, if $n \geq 2$; $\Theta = 0$ otherwise). Then there exists an unique prime dilator F and an unique natural transformation T from F to G such that

$$\mathrm{rg}(T(x)) = \{(\Theta; x_0, \ldots, x_{n-1}; x);\quad x_0 < \cdots < x_{n-1} < x\};$$

obviously $\sigma_F = \sigma$.

Corollary 3.2.6. *There are exactly $n!$ prime dilators of degree n.*

Remarks 3.2.7. (i) If F is prime, then $F(k)$ is a polynomial in k of degree n, where n is the degree of F. $F(k)$ is equal to the number of strictly increasing sequences $x_0 < \cdots < x_{n-1} < k$, that is the binomial coefficient $k(k-1)\cdots(k-n+1)/n!$.

(ii) The infinite value $F(\omega)$ depends not only on the degree n of F, but also on σ_F. The reader can verify that $F(\omega) = \omega^k$, where k is the greatest integer such that one can find a sequence

$$i_0 = 0 < i_1 < \cdots < i_{k-1}, \quad \text{with} \quad \sigma(i_0) < \cdots < \sigma(i_{k-1}) = n-1.$$

(iii) The following property was used implicitly in the section: If F is a dilator, if $z \in F(x)$, then there exists an unique prime dilator G and an unique natural transformation T from G to F such that $z \in \mathrm{rg}(T(x))$. (Indeed, define, if $z = (z_0: x_0, \ldots, x_{n-1}: x)$, the sets $\mathrm{rg}(T(y))$ by

$$\mathrm{rg}(T(y)) = \{(z_0; y_0, \ldots, y_{n-1}; y);\ y_0 < \cdots < y_{n-1} < y\}, \ldots)$$

Definition 3.2.8. Let F be a dilator, and assume that $z_0 = (z_0; 0, \ldots, n-1; n)_F$. Define a prime dilator G and a natural transformation T from G to F as in Remark 3.2.7(iii), with $z = z_0$, $x = n$, then one defines the permutation $\sigma_{z_0,n}$ of n to be σ_G.

Proposition 3.2.9. *Assume that F is a perfect dilator $\neq \underline{1}$, and that $u = (u_0; x_0, \ldots, x_{m-1}; x)$, $v = (v_0; y_0, \ldots, y_{n-1}; x)$, and let $\sigma = \sigma_{u_0,m}$, $\tau = \sigma_{v_0,n}$, then $u < v \to x_{\sigma(0)} \leq y_{\tau(0)}$.*

Proof. Let $p = \sigma(0)$, $q = \tau(0)$, and assume for contradiction that $u \leq v$ and $x_p > y_q$.

(1) We first make the following hypothesis: x is limit and $x_p \geq y_q + \omega$. We have:

$$(u_0; x_0, \ldots, x_{p-1}, x_p, x_p + 1, \ldots, x_p + m - 1 - p; x) \leq u,$$

and also, by Theorem 3.2.4,

$$(v_0; y_0, \ldots, y_{q-1}, y_q + 1, \ldots, y_q + n - q; x) \geq v.$$

Since $x_p > y_q + n - q$, if follows from Proposition 2.3.17 that the values

$$U(z) = (u_0; x_0, \ldots, x_{p-1}, z, z+1, \ldots, z + m - 1 - p; x)$$

are bounded by $(v_0; y_0, \ldots, y_{q-1}, y_q + 1, \ldots, y_q + n - q; x)$.

But the values $(u_0; z_0, \ldots, z_{m-1}; x)$ when $z_0, \ldots, z_{m-1}$ vary through strictly increasing sequences of ordinals $<x$ and of length m, are cofinal in $F(x)$, because F is perfect (immediate from the definition of $\equiv$). But, by Theorem 3.2.4, $(u_0; z_0, \ldots, z_{m-1}; x) \leq U(z_p+1)$, contradiction.

(2) In the general case, replace u and v by their images under $F(f)$, with $f = E_{1\omega} \cdot E_x \in I(x, \omega, x)$; (1) applied to the images yields $f(x_p) \leq f(y_q)$, hence $x_p \leq y_q$.

Corollary 3.2.10. *Assume that F is perfect dilator$\neq \underline{1}$, such that, if $z_0 = (z_0; 0, \ldots, n-1)$, then $\sigma_{z_0,n}(0) = n-1$, for all z_0 and n; then F is a flower.*

Proof. We show that $F(E_{xy}) = E_{F(x)F(y)}$, i.e., that $\mathrm{rg}(F(E_{xy}))$ is an initial segment of $F(y)$: we show that for all $z, z' \in F(y)$ with $z \leq z'$, if $z' \in \mathrm{rg}(F(E_{xy}))$, then $z \in \mathrm{rg}(F(E_{xy}))$. Let $z = (z_0; x_0, \ldots, x_{m-1}; y)$, $z' = (z'_0; x'_0, \ldots, x'_{n-1}; y)$, $\sigma = \sigma_{z_0,m}$, $\tau = \sigma_{z'_0,n}$. By Proposition 3.2.9, if $z \leq z'$, then $x_{\sigma(0)} \leq x'_{\tau(0)}$, i.e., $x_{m-1} \leq x'_{n-1}$, but $z' \in \mathrm{rg}(F(E_{xy}))$, hence $x'_{n-1} < x$, so $x_{m-1} < x$, that is $z \in \mathrm{rg}(F(E_{xy}))$.

Proposition 3.2.11. *If F is a flower, if $z = (z_0; x_0, \ldots, x_{n-1})$, with $n \neq 0$, then $\sigma_{z_0,n}(0) = n-1$.*

Proof. Define $a = (z_0; 0, \ldots, p-1, p+1, \ldots, n)$ and $b = (z_0; 0, \ldots, n-2, n+1)$, with $p = \sigma_{z_0,n}(0)$. If $p < n-1$, then (Theorem 3.2.4) we have $b < a$, but $a \in \mathrm{rg}(F(E_{n+1n+2}))$, whereas $b \notin \mathrm{rg}(F(E_{n+1n+2}))$, contradiction.

Remark 3.2.12. All these properties concerning permutations can be obtained directly from the results of Section 6.4.

3.3. Perfect dilators

Proposition 3.3.1. *If $F \neq \underline{0}$, then the following are equivalent:*

(i) *F is perfect.*

(ii) *The points $F(f)(0)$, when f varies through $I(\omega, \omega)$ are cofinal in $F(\omega)$.*

Proof. F perfect$\leftrightarrow F$ has only one equivalence class modulo $\equiv \leftrightarrow$ all points (ω, z) are $\equiv$-equivalent$\leftrightarrow$

$$\forall z, z' < F(\omega)\, \exists f \in P(\omega, \omega) \quad \text{with } z \leq F(f)(z') \leftrightarrow$$

$$\forall\, z < F(\omega)\, \exists\, f \in I(\omega, \omega) \quad \text{with } F(f)(0) \geq z.$$

Corollary 3.3.2. *If F is perfect and $\neq \underline{1}$, then*

(i) *$F(\omega)$ is limit.*

(ii) $F(0) = 0$.

Proof. Write the points $F(f)(0)$ in the F-representation, as $(0; f(0), \ldots, f(k-1); \omega)$; $k \neq 0$, because these points are cofinal in $F(\omega)$, and $F(\omega) \neq 1$. From this it is easy to conclude that $(0; n, \ldots, n+k-1; \omega) = s_n$ defines a strictly increasing cofinal sequence in $F(\omega)$. So $F(\omega)$ is limit. Suppose that $F(0) \neq 0$, and let $a = F(E_{0\omega})(0)$, then, if $f \in I(\omega, \omega)$,

$$F(f)(0) \leq F(f)(a) = F(fE_{0\omega})(0) = F(E_{0\omega})(0) = a.$$

So the values $F(f)(0)$ cannot be cofinal in the limit ordinal $F(\omega)$.

Remark 3.3.3. The proof of Corollary 3.3.2(i) shows clearly that $F(x)$ is limit for all x limit. So this property characterizes dilators of types ω or Ω.

Definition 3.3.4. Let F be a perfect dilator, and assume that $F \neq \underline{1}$. If x, y are ordinals, we define a subset $\overline{F(x, y)}$ of $F(y+x)$ as follows: $z = (z_0; x_0, \ldots, x_{n-1}; y+x) \in \overline{F(x, y)}$ iff (with $\sigma = \sigma_{z_0, n}$; remark that $n > 0$)

(i) $x_{\sigma(0)} < y$,

(ii) if $\sigma(0) \neq n-1$, then $y \leq x_{\sigma(0)+1}$.

Theorem 3.3.5. *If $f \in I(x, x')$, if $g \in I(y, y')$, then $F(g+f)$ maps $\overline{F(x, y)}$ into $\overline{F(x', y')}$.*

Proof. Immediate from Remark 2.3.13(iii).

Theorem 3.3.6. *If F and G are perfect dilators $\neq \underline{1}$. If T is a natural transformation from F to G, then $T(y+x)$ maps $\overline{F(x, y)}$ into $\overline{G(x, y)}$.*

Proof. Immediate from Proposition 2.3.15.

Definition 3.3.7. (i) Suppose that F is a perfect dilator $\neq \underline{1}$, then define $F(x, y) =$ order type of $\overline{F(x, y)}$, and, if $f \in I(x, x')$, $g \in I(y, y')$; $F(f, g)(t) = u$ iff the tth element of $\overline{F(x, y)}$ is sent by $F(g+f)$ on the uth element of $\overline{F(x', y')}$.

(ii) Moreover, suppose that G is another perfect dilator $\neq \underline{1}$, and that T is a natural transformation from F to G, then define $T(x, y) \in I(F(x, y), G(x, y))$ by $T(x, y)(t) = u$ iff the tth element of $\overline{F(x, y)}$ is sent by $T(y+x)$ on the uth element of $\overline{G(x, y)}$.

Theorem 3.3.8. (i) *$F(\cdot, \cdot)$ is a functor from* ON$\times$ON *to* ON *commuting to direct limits and to pull-backs, $T(\cdot, \cdot)$ is a natural transformation from $F(\cdot, \cdot)$ to $G(\cdot, \cdot)$.*

(ii) *If $x \in$ ON, the unary functor $F(x, \cdot)$, defined by $F(x, y)$, $F(E_x, g)$, is a flower.*

(iii) *There exists an integer n such that for all $x \geq n$, $F(x, \cdot)$ is not a constant functor.*

Proof. (i) The fact that $F(\cdot, \cdot)$ is a functor and that $T(\cdot, \cdot)$ is a natural transformation is obvious from the construction. $F(\cdot, \cdot)$ commutes to $\varinjlim$: if $z \in F(x, y)$,

$z=(z_0; x_0,\dots,x_{n-1}; y+x)$, let $q=\sigma_{z_0,n}(0)+1$, $p=n-q$. Define $f\in I(p,x)$ and $g\in I(q,y)$ by $(g+f)(i)=x_i$. Then

$$(z_0; 0,\dots,n-1; q+p)\in\overline{F(p,q)}\quad\text{and}$$
$$z=F(g+f)(z_0; 0,\dots,n-1; q+p).$$

$\overline{F(\cdot,\cdot)}$ commutes to pull-backs: if $X\subset x$, $Y\subset y$, define $\overline{F(x,X,y,Y)}$ by: $\overline{F(x,X,y,Y)}$ is the image of $F(x,X,y,Y)$ under the function φ from $F(x,y)$ into $F(y+x)$ whose image is $\overline{F(x,y)}$. Obviously

$$\overline{F(x,X,y,Y)}=\overline{F(x,y)}\cap F(y+x,(y,Y)+(x,X)),$$

and from this it is immediate that

$$\overline{F(x,X,y,Y)}\cup\overline{F(x,X',y,Y')}=\overline{F(x,X\cap X',y,Y\cap Y')}.$$

(iii) This is immediate with n such that $F(n)\neq 0$: $\overline{F(n,0)}=\emptyset$, whereas $\overline{F(n,n)}\neq\emptyset$.

(ii) The proof follows the proof of Corollary 3.2.10. Assume that $z,z'\in\overline{F(x,y')}$, $z\leqslant z'$, and $z'\in\operatorname{rg}(F(E_{yy'}+E_x))$. We show that $z\in\operatorname{rg}(F(E_{yy'}+E_x))$ (this will prove that $F(E_x,E_{yy'})=E_{F(x,y)F(x,y')}$). Write

$$z=(z_0; x_0,\dots,x_{m-1}; y'+x),\qquad z'=(z'_0; x'_0,\dots,x'_{n-1}; y'+x).$$

Let $\sigma=\sigma_{z_0,m}$, $\tau=\sigma_{z'_0,n}$. By Proposition 3.2.9, we get $x_{\sigma(0)}\leqslant x'_{\tau(0)}$. Since $z,z'\in\overline{F(x,y')}$, we have the equivalences:

$$z\in\operatorname{rg}(F(E_{yy'}+E_x))\leftrightarrow x_{\sigma(0)}<y$$

and

$$z'\in\operatorname{rg}(F(E_{yy'}+E_x))\longleftrightarrow x'_{\sigma(0)}<y,\quad\text{so}$$
$$z'\in\operatorname{rg}(F(E_{yy'}+E_x))\to z\in\operatorname{rg}(F(E_{yy'}+E_x)).$$

Examples 3.3.9. (i) If F is a perfect flower $\neq\underline{1}$, then $F(x,y)=\overline{F(x,y)}=F(y)$, $\overline{F(f,g)}=F(g)$. This follows from the following remark: by Proposition 3.2.11, $\overline{F(x,y)}=\operatorname{rg}(F(E_{yy+x}))=F(y)$, and it is immediate that $F(x,y)=F(y)$, $F(f,g)=F(g)$.

(ii) Suppose that $F=\mathrm{Id}\cdot\mathrm{Id}=\mathrm{Id}^2$, then

$$F(x,0)=0,\qquad F(f,E_0)=E_0,\tag{1}$$
$$F(x,y+1)=F(x,y)+y+1+x,\qquad F(f,g+E_1)=F(f,g)+g+E_1+f,\tag{2}$$
$$F(f,g+E_{01})=F(f,g)+E_{0y'+1+x'},\tag{3}$$

when $f\in I(x,x')$, $g\in I(y,y')$

$$F(x,\sup(y_i))=\sup(F(x,y_i)),\qquad F(f,\textstyle\bigcup g_i)=\bigcup F(f,g_i).\tag{4}$$

We prove (1) and (2) ((3) and (4) hold in general: they are consequences of property (FL) for y and of commutation to $\varinjlim$).

Let $z = xu + v$, $F(x) = x^2$, then
–if $u < v$, $xu + v = (1; u, v; x)$, $\sigma_{1,2}(0) = 0$, $\sigma_{1,2}(1) = 1$;
–if $u = v$, $xu + v = (0; u; x)$, $\sigma_{0,1}(0) = 0$;
–if $v < u$, $xu + v = (2; v, u; x)$, $\sigma_{2,2}(0) = 1$, $\sigma_{2,2}(1) = 0$.
It is immediate that $\overline{F(x, 0)} = \emptyset$; this establishes (1).

$$\overline{F(x, y)} = \{(0; u; y + x); u < y\} \cup \{(1; u, v; y + x); u < y \leq v < y + x\}$$
$$\cup \{(2; v, u; y + x); v < u < y\}.$$

If $z \in \overline{F(x, y)}$, then $F(E_{yy+1} + E_x)(z) \in \overline{F(x, y + 1)}$, If $z \in \overline{F(x, y + 1)}$, but $z \notin \mathrm{rg}(F(E_{yy+1} + E_x))$, one sees easily that $z = (y + 1 + x)y + v$, with $v < y + 1 + x$. Conversely all points $z = (y + 1 + x)y + v$ are in $\overline{F(x, y + 1)}$, but not in $\mathrm{rg}(F(E_{yy+1} + E_x))$. This shows that $F(x, y + 1) = F(x, y) + y + 1 + x$. Let $f \in I(x, x')$, $g \in I(y, y')$. If $z \in \overline{F(x, y + 1)} \cap \mathrm{rg}(F(E_{yy+1} + E_x))$, then

$$F(g + E_1 + f)(z) = F(g + E_1 + f)F(E_{yy+1} + E_x)(t)$$
$$= F(E_{y'y'+1} + E_{x'})F(g + f)(t).$$

If $z = (y + 1 + x)y + v$, then

$$F(g + E_1 + f)(z) = (y' + 1 + x')y' + (g + E_1 + f)(v).$$

Hence $F(f, g + E_1) = F(f, g) + g + E_1 + f$.

The results of Section 3.6 will enable us to prove similar results directly.

(iii) There exists a dilator F such that $F(x, y) = x \cdot y$, $F(f, g) = f \cdot g$. F is the prime dilator corresponding to the permutation σ of 2 defined by $\sigma(0) = 0$, $\sigma(1) = 1$.

$$(0; x_0, x_1; y + x) \in \overline{F(x, y)} \quad \text{iff } x_0 < y \leq x_1.$$

By Theorem 3.2.4

$$(0; x_0, x_1; y + x) < (0; x_0', x_1'; y + x) \quad \text{iff } x_0 < x_0' \quad \text{or} \quad x_0 = x_0' \quad \text{and} \quad x_1 < x_1'.$$

So

$$\overline{F(x, y + 1)} = \{(0; x_0, x_1; y + 1 + x); x_0 < y + 1 \leq x_1 < y + 1 + x\}$$

implies that $F(x, y + 1) = F(x, y) + x$, and it is immediate that

$$F(x, 0) = 0, \qquad F(f, E_0) = E_0,$$
$$F(x, y + 1) = F(x, y) + x, \qquad F(f, g + E_1) = F(f, g) + f,$$

if $f \in I(x, x')$, $g \in I(y, y')$

$$F(f, g + E_{01}) = F(f, g) + E_{0x'},$$
$$F(x, \sup(y_i)) = \sup F(x, y_i), \qquad F(f, \bigcup g_i) = \bigcup F(f, g_i).$$

But this defines exactly the functor product. Once again, this can be established directly from the general results of Section 3.6.

3.4. *Dilators of kind Ω and bilators*

Definition 3.4.1. The following data define a category ΩDIL:
objects: dilators of kind Ω;

morphisms from $F = F' + F''$ *to* $G = G' + G''$: (F'', G'' perfect) the set $\Omega I(F, G)$ of natural transformations $T = T' + T''$, when T' is a natural transformation from F' to G', T'' is a natural transformation from F'' to G''.

Definition 3.4.2. The following data define a category BIL·

objects: *bilators*, i.e., functors from ON×ON to ON such that:

(i) F commutes to $\varinjlim$ and &;

(ii) for all $x \in$ ON, the partial functor F_x (see Examples 2.4.5) enjoys (FL), i.e., is a flower;

(iii) $F(x, y)$ is not constant in y, i.e., F cannnot be put in the form: $F(x, y) = G(x)$, $F(f, g) = G(f)$ for some G.

morphisms from F *to* G: the set $I(F, G)$ of natural transformations from F to G.

Remarks 3.4.3. (i) The simplest examples of bilators are the sum, the product and the exponential, as in Remark 2.4.6. Since we shall see that the categories BIL and ΩDIL are isomorphic, it is not necessary to produce more examples of bilators.

(ii) Warning: the categories BIL and ΩDIL do not enjoy the existence of pull-backs; we give an example in ΩDIL. Let $F = \mathrm{Id}^2$, and define prime dilators F_1 and F_2, together with $T_i \in \Omega I(F_i, F)$, by $i \in \mathrm{rg}(T_i(2))$, then T_1 & T_2 does not exist. (But it exists in the larger category DIL, and is equal to E_{0F}; pull-backs in categories of dilators are studied in Section 4.)

Remarks 3.4.4. (i) $I(F, G)$ and $\Omega I(F, G)$ are sets because natural transformations from F to G are determined by their restrictions to $\mathrm{ON}{<}\omega \times \mathrm{ON}{<}\omega$ or to $\mathrm{ON}{<}\omega$.

(ii) In the definition of bilators, condition (iii) can be replaced by: $F(n, m) \neq F(n, m+1)$ for some integers n and m: if F is non constant in y, then the partial functors $F(n, y)$, $F(E_n, g)$ cannot be constant for all n (direct limit argument). If the flower $F(n, y)$, $F(E_n, g)$ is non constant, then $F(n, m) \neq F(n, m+1)$ for some m by Remark 2.4.10(ii).

(iii) If F is a bilator, then it is possible to represent ordinals by means of sequences $(z_0; x_0, \ldots, x_{n-1}; x; y_0, \ldots, y_{m-1}) = z$. This means that $z = F(f, g)(z_C)$ with $f \in I(n, x)$, $g \in I(m, y)$ for some $y > y_{m-1}$, and that m and n are both minimum for this property. (See Theorem 2.3.12, Remarks 2.3.13 and 2.4.8.)

Definition 3.4.5. We define the functor SEP (*separation of variables*) by:

$$\mathrm{SEP}(F' + F'')(x, y) = F'(x) + F''(x, y),$$

$$\mathrm{SEP}(F' + F'')(f, g) = F'(f) + F''(f, g),$$

$$\mathrm{SEP}(T' + T'')(x, y) = T'(x) + T''(x, y),$$

when F'', G'' are perfect, $T'' \in \Omega I(F'', G'')$, $T' + T'' \in \Omega I(F' + F'', G' + G'')$. $F''(x, y)$, $G''(x, y)$, $T''(x, y)$ have been defined in Definition 3.3.7.

SEP is a functor from ΩDIL to BIL.

Definition 3.4.6. If F is a bilator, if $x \in \mathrm{ON}$, then define $\overline{F(x)} \subset F(x, x)$ by:

$$(z_0; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1}) \in \overline{F(x)}$$
$$\leftrightarrow (nm = 0 \vee (nm \neq 0 \wedge y_{n-1} < x_0)).$$

Theorem 3.4.7. (i) *If F is a bilator and $f \in I(x, y)$, then $F(f, f)$ maps $\overline{F(x)}$ into $\overline{F(y)}$.*
(ii) *If F and G are bilators, if $T \in I(F, G)$, then $T(x, x)$ maps $\overline{F(x)}$ into $\overline{G(x)}$.*

Proof. This immediate theorem is left to the reader.

Definition 3.4.8. One defines the functor UN (*unification of variables*) by:

$\mathrm{UN}(F)(x) =$ order type of $\overline{F(x)}$,

$\mathrm{UN}(F)(f)(z) = z'$ iff the image under $F(f, f)$ of the zth element of $\overline{F(x)}$ is the z'th element of $\overline{F(y)}$ $(f \in I(x, y))$,

$\mathrm{UN}(T)(x)(z) = z'$ iff the image under $T(x, x)$ of the zth element of $\overline{F(x)}$ is the z'th element of $\overline{G(x)}$ $(F, G$ bilators, $T \in I(F, G))$.

Theorem 3.4.9. UN *is a functor from* BIL *to* ΩDIL.

Proof. It is immediate that $\mathrm{UN}(F)$ is a functor from ON to ON, and that $\mathrm{UN}(T)$ is a natural transformation from $\mathrm{UN}(F)$ to $\mathrm{UN}(G)$.

$\mathrm{UN}(F)$ commutes to $\varinjlim$: it suffices to show that, if $z \in \overline{F(x)}$, then there exist $p, z' < \overline{F(p)}$ and $f \in I(p, x)$ such that $z = F(f, f)(z')$. Write

$$z = (z_0; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1})$$

and let $p = n + m$, and f be defined by

$$\mathrm{rg}(f) = \{y_0, \ldots, y_{n-1}, x_0, \ldots, x_{m-1}\},$$

then

$$z = F(f, f)(z_0; n, \ldots, n+m-1; n+m; 0, \ldots, n-1).$$

$\mathrm{UN}(F)$ commutes to &: if $X \subset x$, define $\overline{F(x, X)}$ by: $\overline{F(x, X)}$ is the image of $\mathrm{UN}(F)(x, X)$ under the function φ from $\mathrm{UN}(F)(x)$ to $F(x, x)$ whose range is $\overline{F(x)}$. It is immediate that $\overline{F(x, X)} = \overline{F(x)} \cap F(x, X; X)$, and from this that $\overline{F(x, X)} \cap \overline{F(x, X')} = \overline{F(x, X \cap X')}$. From this we conclude that $\mathrm{UN}(F)$ commutes to &.

$\mathrm{UN}(F)$ is of kind Ω.

Subcase 1. Assume that $F(\omega, 0) = 0$, and let a_0 be the smallest point in $\overline{F(\omega)}$. We have:

$$F(f, f)(a_0) \geq F(f, f)(0) = F(f, f)(F(\omega, 0)) \geq F(E_\omega, f)(F(\omega, 0)) \geq F(\omega, f(0)),$$

using the fact that, when G is a flower, $G(f)(G(0)) \geq G(f(0))$, which comes easily from Remark 2.4.10(i). Since $F(\omega,\cdot)$ is a flower, we get $F(\omega,\omega)=\sup_n(F(\omega,n))$, and we conclude that the points $F(f,f)(a_0)$ are cofinal in $F(\omega,\omega)$ (hence in $\overline{F(\omega)}$) when f varies through $I(\omega,\omega)$. This means exactly that the points $\mathrm{UN}(F)(f)(0)$ are cofinal in $\mathrm{UN}(F)(\omega)$. So, by Proposition 3.3.1 that $\mathrm{UN}(F)$ is perfect. Remark that $\mathrm{UN}(F)$ is non constant, so $\mathrm{UN}(F)\neq \underline{1}$: $\mathrm{UN}(F)$ is of kind Ω.

Subcase 2. If $F(\omega,0)\neq 0$, then observe that the point

$$z=(z_0;x_0,\ldots,x_{n-1};x;y_0,\ldots,y_{m-1})$$

is strictly smaller that $F(x,0)$ iff $m=0$, because $F(x,\cdot)$ is a flower (Remark 2.4.10(i)). So it is possible to write

$$F(x,y)=F(x,0)+F''(x,y),\qquad F(f,g)=F(f,E_0)+F''(f,g).$$

Let $F'(x)=F(x,0)$, $F'(f)=F(f,E_0)$. It is immediate that $F''(\omega,0)=0$, and that $F(x,0)\subset\overline{F(x)}$. More precisely, if $F(x,0)\leqslant z<F(x,x)$, then

$$\begin{aligned}z&=(z_0;x_0,\ldots,x_{n-1};x;y_0,\ldots,y_{m-1})_F\\&=F(x,0)+(a,x_0,\ldots,x_{n-1};x;y_0,\ldots,y_{m-1})_{F''}\end{aligned}$$

with $z_0=F(n,0)+a$, hence $z=F(x,0)+b\in\overline{F(x)}$ iff $b\in\overline{F''(x)}$.

Once concludes that $\mathrm{UN}(F)=F'+\mathrm{UN}(F'')$. By subcase one, $\mathrm{UN}(F)$ is of kind Ω. $\mathrm{UN}(T)\in\Omega I(\mathrm{UN}(F),\mathrm{UN}(G))$: Write $F=F'+F''$, $G=G'+G''$, with $F''(\omega,0)=0$, $G''(\omega,0)=0$, hence one may write $T=T'+T''$, with $T''\in I(F,G)$, then $\mathrm{UN}(T)=T'+\mathrm{UN}(T'')$. So $\mathrm{UN}(T)\in\Omega I(\mathrm{UN}(F),\mathrm{UN}(G))$.

Remark 3.4.10. Of course the functor Δ defined by: $\Delta(F)(x)=F(x,x)$, $\Delta(F)(f)=F(f,f)$, $\Delta(T)(x)=T(x,x)$, is a more natural 'unification of variables', but this functor is not inversible. Remark that there is a natural transformation from UN to Δ, defined by: $\Theta(F)(x)(z)=$ the zth point of $\overline{F(x)}$.

Theorem 3.4.11. (i) $\mathrm{SEP}\circ\mathrm{UN}=\mathrm{ID}_{\mathrm{BIL}}$.
(ii) $\mathrm{UN}\circ\mathrm{SEP}=\mathrm{ID}_{\Omega\mathrm{DIL}}$.

Proof. (i) *Subcase* 1. Assume that $F(\omega,0)=0$, then we show that $\mathrm{SEPUN}(F)=F$. We know that $G=\mathrm{UN}(F)$ is a perfect dilator, and that $T=\Theta(F)$ (Remark 3.4.10) is a natural transformation from $\mathrm{UN}(F)$ to $\Delta(F)=H$. Define the binary functor K by: $K(x,y)=H(y+x)$, $K(f,g)=H(g+f)$, then (H is perfect) define a natural transformation U from $\mathrm{SEP}(H)$ to K, by: $U(x,y)(z)=$ the zth element of $\overline{H(x,y)}$; then $V=U\,\mathrm{SEP}(T)$ is a natural transformation from $\mathrm{SEPUN}(F)$ to K:

$$\begin{tikzcd}
\mathrm{UN}(F)(x)=G(x) \arrow[d, "T(x)"] & & \mathrm{SEPUN}(F)(x,y)=G(x,y) \arrow[d, "T(x,y)"] \arrow[dd, bend right, "V(x,y)"'] \\
F(x)=H(x)=F(x,x) & & H(x,y) \arrow[d, "U(x,y)"] \\
& & K(x,y)=H(y+x)=F(y+x,y+x)
\end{tikzcd}$$

Now, we compute the sets $\mathrm{rg}(V(x,y)) \subset F(y+x, y+x)$: $z \in \mathrm{rg}(V(x,y))$ iff $z = U(x,y)(z')$ for some $z' \in \mathrm{rg}(T(x,y)$; but $z' \in \mathrm{rg}(T(x,y))$ iff $z' = T(x,y)(u)$ for some u, i.e., the uth element of $\overline{\mathrm{UN}(F)(x,y)}$ is sent by $T(y+x)$ on the z'th element of $\overline{H(x,y)}$. But, if $z = U(x,y)(z')$, then z is precisely the z'th element of $\overline{H(x,y)}$. This shows that $\mathrm{rg}(V(x,y)) = \overline{H(x,y)} \cap \mathrm{rg}(T(y+x))$. We study now the set $\mathrm{rg}(T(y+x))$: $z \in \mathrm{rg}(T(y+x))$ iff $z \in \overline{F(y+x)}$. Hence $z \in \mathrm{rg}(T(y+x))$ iff z can be written

$$z = (z_0; x_0, \ldots, x_{m-1}; y+x; y_0, \ldots, y_{n-1})_F, \quad \text{with } m=0 \quad \text{or} \quad y_{n-1} < x_0$$

($n = 0$ is impossible, because $F(\omega, 0) = 0$). One considers the points

$$a_k = \begin{cases} (z_0; 2n, \ldots, 2(n+m-1); 2(n+m); 0, \ldots, 2(k-1), \\ \qquad\qquad 2k+1, 2k+2, \ldots, 2(n-1))_F & \text{if } k < n, \\ (z_0; 2n, \ldots, 2(k-1), 2k+1, 2k+2, \ldots, 2(n+m-1); \\ \qquad\qquad 2(n+m); 0, \ldots, 2(n-1))_F & \text{if } n \leqslant k < n+m. \end{cases}$$

These points are pairwise distinct, and a_{n-1} is the greatest of them all (if $k \neq n-1$, then $a_k < F(2(n+m), 2n-1)$, and $a_{n-1} \geqslant F(2(n+m), 2n-1)$, by Remark 2.4.10(i)). Hence, if

$$z = (z_0; x_0, \ldots, x_{m-1}; y+x; y_0, \ldots, y_{n-1})_F = T(y+x)(z'),$$

then

$$z' = (z_0'; y_0, \ldots, y_{n-1}, x_0, \ldots, x_{m-1}; y+x)_G$$

and

$$\sigma^G_{z_0', n+m}(0) = n-1 = \sigma^H_{z_0, n+m}(0).$$

From this, we conclude that, provided $z \in \mathrm{rg}(T(x,y))$, then $z \in \overline{H(x,y)}$ iff ($y_{n-1} < y$ and ($m = 0$ or $y \leqslant x_0$)), and from this: $z \in \mathrm{rg}(V(x,y))$ iff ($y_{n-1} < y$ and ($m = 0$ or $y \leqslant x_0$)).

Now, consider the natural transformation W from F to K defined by: $W(x,y) = F(E_{0y} + E_x, E_{yy+x})$, then $\mathrm{rg}(W(x,y)) = \mathrm{rg}(V(x,y))$ for all x, y. This proves that $V = W$, so $\mathrm{SEPUN}(F) = F$.

Subcase 2. If $F(\omega, 0) \neq 0$, write $F = F' + F''$, as in the proof of Theorem 3.4.9, with $F''(\omega, 0) = 0$. Since $F''(\omega, 0) = 0$, $\mathrm{SEPUN}(F'') = F''$, and remark that

$$\mathrm{SEPUN}(F' + F'') = \mathrm{SEP}(F' + \mathrm{UN}(F'')) = F' + \mathrm{SEPUN}(F'') = F' + F''.$$

It remains now to show, that if $R \in I(F, G)$, then $\mathrm{SEPUN}(R) = R$. Writing $R = R' + R''$, one reduces to the case $F(\omega, 0) = G(\omega, 0) = 0$. Then it is immediate to see that: $\mathrm{SEPUN}(R)(x,y)(z) = z'$ iff the image under $R(y+x, y+x)$ of the zth element of $\mathrm{rg}(V(x,y))$($= \mathrm{rg}(F(E_{0y} + E_x, E_{yy+x}))$) is the z'th element of

$$\mathrm{rg}(G(E_{0y} + E_x, E_{yy+x});$$

but the diagram

$$\begin{tikzcd}
F(x,y) \arrow[r, "R(x,y)"] & G(x,y) \arrow[d, "G(E_{0y}+E_x,\, E_{yy+x})"] \\
F(y+x,y+x) \arrow[u, "F(E_{0y}+E_x,\, E_{yy+x})"] \arrow[r, "R(y+x,y+x)"'] & G(y+x,y+x)
\end{tikzcd}$$

is commutative, hence we have $R(x, y) = \mathrm{SEPUN}(R)(x, y)$.

(ii) *Subcase* 1. Assume that F is a perfect dilator $\neq \underline{1}$. We show that $\mathrm{UNSEP}(F) = F$. Let G be $\mathrm{SEP}(F)$, and $H(x, y) = F(y+x)$, $H(f, g) = F(g+f)$. Define the natural transformation T from $\mathrm{SEP}(F)$ to H, by $\mathrm{rg}(T(x, y)) = \overline{F(x, y)}$. Let $U = \Theta(\mathrm{SEP}(F))$ be the natural transformation defined in Remark 3.4.10, from $\mathrm{UNSEP}(F)$ to $\Delta(\mathrm{SEP}(F))$. Finally, observe that $\Delta(T)$ is a natural transformation from $\Delta\mathrm{SEP}(F)$ to $K = \Delta(H)$. So $V = \Delta(T)U$ is a natural transformation from $\mathrm{UNSEP}(F)$ to K. K is defined by $K(x) = F(x+x)$, $K(f) = F(f+f)$, i.e., $K = F \circ (\mathrm{Id}+\mathrm{Id})$.

$$\begin{tikzcd}
\mathrm{SEPF}(x,y) = G(x,y) \arrow[d, "T(x,y)"'] \\
H(x,y) = F(y+x)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{UNSEP}(F)(x) \arrow[d, "U(x)"'] \arrow[dd, bend right=60, "V(x)"'] \\
\Delta(\mathrm{SEP}(F))(x) \arrow[d, "\Delta T(x)"'] \\
\Delta H(x) = F(x+x)
\end{tikzcd}$$

$\mathrm{rg}(U(x)) = \overline{\mathrm{SEP}(F)(x)}$. Remark that, if $z \in F(y+x)$, with

$$\begin{aligned} z &= (z_0; x_0, \ldots, x_q, x_{q+1}, \ldots, x_{n-1}; y+x)_F \\ &= (z_0; x'_{q+1}, \ldots, x'_{n-1-q}; x; x_0, \ldots, x_q)_H, \end{aligned}$$

with $x_q < y \leqslant x_{q+1}$, and $y + x'_{q+i} = x_{q+i}$, for $0 < i < n-q$.

We compute the set $\mathrm{rg}(V(x)) \subset F(x+x)$: $z = (z_0; x_0, \ldots, x_{n-1}; x+x)_F \in \mathrm{rg}(V(x))$ iff $z = T(x, x)$ for some $z' \in \overline{G(x)}$. Let $p = \sigma_{z_0,n}(0)$, and define x'_j by $x + x'_j = x_{p+j}$, for $1 \leqslant j \leqslant n-1-p$. Assume that

$$z = (z_0; x'_1, \ldots, x'_{n-1-p}; x; x_0, \ldots x_p) = T(x, x)(u),$$

with

$$u = (u_0; x'_1, \ldots, x'_{n-1-p}; x; x_0, \ldots, x_p)_G.$$

Then $u \in \overline{G(x)}$ iff $p = n-1$ or $x_p < x'_1$, that is $x + x_p < x_{p+1}$. Also, z belongs to $\mathrm{rg}(T(x, x)) = \overline{F(x, x)}$ iff $x_p < x$ and ($p = n-1$ or $x \leqslant x_{p+1}$). Summing up, we get

$$z \in \mathrm{rg}(V(x)) \quad \text{iff } (x_p < x \quad \text{and} \quad (p = n-1 \quad \text{or} \quad x + x_p < x_{p+1})).$$

Now, define the natural transformation W from F to K by:

$$W(x)(z_0; x_0, \ldots, x_{n-1}; x)_F = (z_0; x_0, \ldots, x_p, x + x_{p+1}, \ldots, x + x_{n-1}; x + x)_F$$

with $p=\sigma_{z_0,n}(0)$. The function $W(x)$ is strictly increasing. Assume that

$$z=(z_0;x_0,\ldots,x_{n-1};x)<(z'_0;x'_0,\ldots,x'_{m-1};x)=z',\quad \text{with } q=\sigma_{z'_0,m}(0).$$

By Proposition 2.3.9, $x_p\leqslant x'_q$. If $x_p=x'_q$, then $W(x)(z)=F(f)(z)$, $W(x)(z')=F(f)(z')$, with $f(t)=t$ for $t\leqslant x_p$, $f(t)=x+t$ for $t>x_p$, so $W(x)(z)<W(x)(z')$. If $x_p<x'_q$, then Proposition 2.3.9 forces $W(x)(z)<W(x)(z')$.

Since obviously $\mathrm{rg}(W(x))=\mathrm{rg}(V(x))$, $V=W$, so $\mathrm{UNSEP}(F)=F$.

Subcase 2. F is not perfect. Write $F=F'+F''$, with F'' perfect $\neq \underline{1}$, then

$$\mathrm{UNSEP}(F'+F'')=\mathrm{UN}(F'+\mathrm{SEP}(F''))=F'+\mathrm{UNSEP}(F'')=F'+F''.$$

It remains to show that $\mathrm{UNSEP}(R)=R$, when $R\in\Omega I(F,G)$. Writing $R=R'+R''$, one reduces to the case F,G perfect. If V' is the natural transformation from $\mathrm{UNSEP}(G)=G$ to $G\circ(\mathrm{Id}+\mathrm{Id})=K'$, defined similarily to V, then the diagrams

$$\begin{tikzcd} F(x) \arrow[r, "R(x)"] \arrow[d, "V(x)"'] & G(x) \arrow[d, "V'(x)"] \\ K(x) \arrow[r, "R(x+x)"'] & K'(x) \end{tikzcd}$$

are commutative, but $\mathrm{UNSEP}(R)$ is defined by: $\mathrm{UNSEP}(R)(x)(z)=z'$ iff the image under $R(x+x)$ of the zth element in $\mathrm{rg}(V(x))$ is the z'th element in $\mathrm{rg}(V'(x))$. So $\mathrm{UNSEP}(R)=R$.

Remark 3.4.12. The functors SEP and UN identify perfect dilators $\neq 1$ with bilators satisfying $F(\omega,0)=0$. These bilators will therefore be called *perfect* bilators.

Corollary 3.4.13. *The functors* SEP *and* UN *commute to direct limits and to pull-backs.*

Proof. This is a general property of inversible functors.

Remark 3.4.14. One can imagine other ways of proving Theorem 3.4.11.

(i) Using the results of Section 3.6, it is surely possible to establish that SEP and UN are reciprocal.

(ii) Using dendroids (with, for bilators, multi-dendroids in two colours), one can certainly obtain the result quickly.

Theorem 3.4.15. (i) *If F is a dilator of kind Ω, then*

$$(\Delta\mathrm{SEP}(F))\circ(\underline{\omega}^{1+\mathrm{Id}})=F\circ(\underline{\omega}^{1+\mathrm{Id}}).$$

(ii) *If $T\in\Omega I(F,G)$, then*

$$(\Delta\mathrm{SEP}(T))\circ(\underline{\omega}^{1+\mathrm{Id}})=T\circ(\underline{\omega}^{1+\mathrm{Id}}).$$

Proof. (i) This means that, if F is a bilator, then $F(x,x)=\mathrm{UN}(F)(x)$ for all $x=\omega^{1+x'}$, $F(f,f)=\mathrm{UN}(F)(f)$ for all $f=\underline{\omega}^{1+f}$. Define a function φ_x from $F(x,x)$ to itself by:

$$\begin{aligned}&\varphi_x((z_0;x_0,\ldots,x_{m-1};x;y_0,\ldots,y_{n-1}))\\&\quad=(z_0;y+x_0,\ldots,y+x_{m-1};x;y_0,\ldots,y_{n-1}),\end{aligned}$$

with $y=0$ if $n=0$, $y=y_{n-1}+1$ otherwise. It is immediate that φ_x is strictly increasing, and that $\mathrm{rg}(\varphi_x)=\overline{F(x)}$. From this $F(x,x)=\mathrm{UN}(F)(x)$. In order to prove that $F(f,f)=\mathrm{UN}(F)(f)$, it will therefore be sufficient to show that $\varphi_x\cdot F(f,f)=F(f,f)\varphi_x$ (because $\varphi_x\cdot F(f,f)=\mathrm{UN}(F)(f)\varphi_x$ by definition of $\mathrm{UN}(F)(f)$). But

$$\begin{aligned}&\varphi_x\cdot F(f,f)((z_0;x_0,\ldots,x_{m-1};x;y_0,\ldots,y_{n-1}))\\&\quad=(z_0;y'+f(x_0),\ldots,y'+f(x_{m-1});x';f(y_0),\ldots,f(y_{n-1})),\end{aligned}$$

with $y'=0$ if $n=0$, and $y'=f(y_{n-1})+1$ otherwise. On the other hand, we have

$$\begin{aligned}&F(f,f)\varphi_x((z_0;x_0,\ldots,x_{m-1};x;y_0,\ldots,y_{n-1}))\\&\quad=(z_0;f(y+x_0),\ldots,f(y+x_{m-1});x';f(y_0),\ldots,f(y_{n-1})),\end{aligned}$$

with $y=0$ if $n=0$, and $y=y_{n-1}+1$ otherwise. If $n=0$, then the two expressions are equal. If $n\neq 0$, then we have to show that $f(y_{n-1}+1+x_i)=f(y_{n-1})+1+f(x_i)$. But $f(1)=1$ (because $f=\omega^{1+f'}$), hence it suffices to prove that f is 'linear', i.e., $f(a+b)=f(a)+f(b)$ for all a and b. But this is immediate by looking to the Cantor normal forms of a and b.

(ii) This means that, if F and G are bilators and $T\in I(F,G)$, then $T(x,x)=\mathrm{UN}(T)(x)$. In order to prove this property, it will suffice (since, by definition, $\varphi^G_xT(x,x)=\mathrm{UN}(T)(x)\varphi^F_x$) to show that $\varphi^G_xT(x,x)=T(x,x)\varphi^F_x$. This is left to the reader.

3.5. Induction on dilators

Theorem 3.5.1. *Let P be a property defined on dilators and assume that:*

(i) $P(\underline{0})$.

(ii) $P(F)\to P(F+\underline{1})$.

(iii) *If x is limit, if $F_i\neq\underline{0}$ for all $i<x$, and if for all $y<x$, $P(\sum_{i<y}F_i)$, then $P(\sum_{i<x}F_i)$.*

(iv) *If F is of kind Ω, if for all $y\in\mathrm{ON}$ $P(\mathrm{SEP}(F)^y)$, then $P(F)$ (if G is a bilator, then G^y is the dilator defined by $G^y(x)=G(x,y)$, $G^y(f)=G(f,E_y)$).*

Then $P(F)$ is true for all dilators F.

Proof. Assume that $P(F)$ is false. We define a 'descending sequence' of dilators, by $F_0=F$, and if F_n has been defined and $\neg P(F_n)$:

(i) $F_n\neq\underline{0}$ because of hypothesis (i), so F_n is of kind $1,\omega$, or Ω.

(ii) If $F_n=G+\underline{1}$, then $\neg P(G)$ by (ii), and one may take $F_{n+1}=G$.

(iii) If $F_n = \sum_{i<x} F^i$, with x limit, then, by hypothesis (iii), $P(\sum_{i<y} F^i)$ is false for some $y<x$, hence take $F_{n+1} = \sum_{i<y} F^i$.

(iv) If F_n is of kind Ω, then, by hypothesis (iv), $\neg P(\mathrm{SEP}(F_n)^y)$ for some $y \in \mathrm{ON}$. Take $F_{n+1} = \mathrm{SEP}(F_n)^y$.

Let Y be the set of all ordinals y such that for some n, F_n is of kind Ω and $F_{n+1} = \mathrm{SEP}(F_n)^y$. Let a be an ordinal greater than all points in Y, and such that $a = \omega^{a'}$, with $a' \neq 0$; we show that the values $F_n(a)$ form a strictly decreasing sequence of ordinals, and this will yield a contradiction:

– if $F_n = F_{n+1} + \underline{1}$, then $F_n(a) = F_{n+1}(a) + 1$;

– if $F_n = \sum_{i<x} F^i$, if $F_{n+1} = \sum_{i<y} F^i$, with x limit and $y<x$, then $F^y(a) \neq 0$, because a is infinite and $F^y \neq \underline{0}$, hence $F_n(a) \geq F_{n+1}(a) + F^y(a)$ so $F_n(a) > F_{n+1}(a)$;

– if F_n is of kind Ω, then $F_{n+1} = \mathrm{SEP}(F_n)^y$ for some $y<a$, so $F_{n+1}(a) = \mathrm{SEP}(F_n)(a, y) < \mathrm{SEP}(F_n)(a, a)$ (by Remark 2.4.10(ii) since a is infinite). But by Remark 3.4.15, $\mathrm{SEP}(F_n)(a, a) = \mathrm{UNSEP}(F_n)(a) = F_n(a)$, so, again $F_{n+1}(a) < F_n(a)$.

Remark 3.5.2. The hypotheses (i)–(iv) indicate clearly which dilators must be considered as the *predecessors* of F: G is a predecessor of F iff there is a finite sequence $F_0 = G, \ldots, F_n = F$, such that, for all $i<n$, $F_{i+1} = F_i + F_i'$ for some $F_i' \neq \underline{0}$, or $F_i = \mathrm{SEP}(F_{i+1})^y$ for some $y \in \mathrm{ON}$. (The condition $F_{i+1} = F_i + F_i'$ corresponds to the iteration of clauses (ii) and (iii) in Theorem 3.5.1.) The meaning of Theorem 3.5.1 is that the predecessor relation is well founded. This relation is not a linear order, but, given F, the 'set' of predecessors of F is linearily ordered by the predecessor relation. There are many other well-founded predecessor relations that can be defined on dilators, for instance say that in a sum $F = \sum_{i<x} F_i$, with $x \geq 2$ and $F_i \neq \underline{0}$ for all i, then F_i is a predecessor of F. However, such a definition (which is more in the spirit of dendroids) is such that the 'set' of predecessors of F is not linearily ordered.

Remark 3.5.3. It is clear that the 'set' of predecessors of F, ordered by the predecessor relation is a linear well-order, so it is isomorphic to an 'ordinal class', i.e., an ordinal which is a class, for instance, if $F = \mathrm{Id}$, then this ordinal class is exactly ON. One proves easily that the order type of the class of predecessors of F has order type $F(\mathrm{ON})$. (The ordinal class $F(\mathrm{ON})$ is defined to be $\varinjlim^*_{x \in \mathrm{ON}}(F(x), F(E_{xy}))$, the limit being taken on a class (ON) instead of a set, as usual.) (Hint: prove that the predecessor G of F has order type $G(\mathrm{ON})$, and use Theorem 8.4.15 and $\omega^{\mathrm{ON}} = \mathrm{ON}$.)

Remark 3.5.4. The most remarkable feature of Theorem 3.5.1 is that it can be applied to construct functors by induction on dilators. The most important case, up to now is $\bigwedge$; see Section 5.

Remark 3.5.5. We shall see in part II that induction on dilators is the same thing as traditional bar induction of type 2, and, of course, that recursion on dilators is bar-recursion of type 2.

3.6. An alternative description of UN *and* SEP

Definition 3.6.1. (i) If F is a bilator, then define ${}^yF$ for all $y \in \mathrm{ON}$ by:

$$F(x, y) + {}^yF(x) = F(x, y+1),$$

$$F(f, E_y) + {}^yF(f) = F(f, E_{y+1}) \qquad (\text{i.e., } F^y + {}^yF = F^{(y+1)}).$$

(ii) If F is a bilator, then define, for all $g \in I(y, y')$, a natural transformation ${}^gF$ from ${}^yF$ to ${}^{y'}F$ by:

$$F(E_x, g) + {}^gF(x) = F(E_x, g + E_1)$$

(if one adopts, when T is a natural transformation from F to G, and $f \in I(x, x')$ the notation $T(f)$ for $G(f)T(x) = T(x')F(f)$, then we have $F(f, g) + {}^gF(f) = F(f, g + E_1)$).

(iii) If F, G are bilators, if T is a natural transformation from F to G, then define for all $y \in \mathrm{ON}$ a natural transformation ${}^yT$ from y_F to ${}^yG$ by:

$$T(x, y) + {}^yT(x) = T(x, y+1) \qquad (\text{hence } T(f, E_y) + {}^yT(f) = T(f, E_{y+1})).$$

Theorem 3.6.2. (i) *Suppose that F is a bilator, then* UN(F) *is the functor G defined as follows:*

$$G(x) = F(x, 0) + \sum_{y<x} {}^yF(x - (y+1)),$$

$$G(f) = F(f, E_0) + \sum_{y<x'} h_y, \quad \textit{when } f \in I(x, x'),$$

and with

$$h_y = \begin{cases} E_{0^yF(x'-(y+1))} & \textit{if } y \notin \mathrm{rg}(f), \\ {}^{(f_y)}F(x'-(y+1))^zF(f^y) = {}^{(f_y)}F(f^y), & \textit{if } y \in \mathrm{rg}(f), \quad \textit{with } z = f^{-1}(y), \end{cases}$$

and $f_y \in I(z, y)$, $f^y \in I(x - (z+1), x' - (y+1))$ *are such that* $f = f_y + E_1 + f^y$.

(ii) *If F and G are bilators, if* $T \in I(F, G)$, *then* UN(T) *is the natural transformation U defined by:*

$$U(x) = T(x, 0) + \sum_{y<x} {}^yT(x - (y+1)).$$

Proof. (i) If $z < F(x, 0)$, then $z = (z_0; x_0, \ldots, x_{k-1}; x;)$, hence $z \in \overline{F(x)}$, and $F(f, f)(z) = F(f, f)\underline{E_{F(x,0)F(x,x)}(z)} = F(f, E_0)(z)$. Assume now that $F(x, y) \leq z < F(x, y+1)$ and that $z \in \overline{F(x)}$, then

$$z = (z_0; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-2}, y).$$

If $F(x - (y+1), y) \leq t < F(x - (y+1), y+1)$, then

$$t = (t_0; x'_0, \ldots, x'_{p-1}; x - (y+1); y'_0, \ldots, y'_{q-1}, y).$$

Define $g \in I(x-(y+1))$ by $g(u) = y+1+u$, then $F(g, E_{y+1})$ induces a strictly increasing bijection between the sets $[F(x-(y+1), y), F(x-(y+1), y+1)[$ and $[F(x, y), F(x, y+1)[\cap \overline{F(x)}$. Hence, the order type of $[F(x, y), F(x, y+1)[\cap \overline{F(x)}$ is equal to ${}^{y}F(x-(y+1))$. From this we deduce that

$$G(x) = F(x, 0) + \sum_{y<x} F(x-(y+1)).$$

Assume now that

$$z = F(g, E_{y+1})(F(x-(y+1), y) + u), \quad \text{with } u < {}^{y}F(x-(y+1)),$$

and let $w = F(x-(y+1), y) + u$, $g \in I(x-(y+1), x)$, $h \in I(x'-(f(y)+1), x')$ defined by $g(a) = y+1+a$, $h(a) = f(y)+1+a$; then

$$F(f, f)(z) = F(h, E_{f(y)+1})(F(x'-(f(y)+1), f(y)) + v),$$

and it will suffice to prove that $v = h_{f(y)}(u)$, to end the proof of (i).

If $t < x-(y+1)$, then

$$fg(t) = f(y+1+t) = f(y)+1+f^{y}(t) = hf^{y}(t),$$

hence

$$fg = hf^{y} \cdot F(f, f)(z) = F(f, f)(F(g, E_{y+1})(w)) = F(fg, fE_{y+1x})(w)$$
$$= F(hf^{y}, f_y + E_1)(w) = F(h, E_{f(y)+1})F(f^{y}, f_y + E_1)(w),$$

hence

$$F(f^{y}, f_y + E_1)(w) = F(x'-(f(y)+1), f(y)) + v,$$

and we have:

$$\begin{aligned} &F(f^{y}, f_y + E_1)(w) \\ &= F(E_{x'-(f(y)+1)}, f_y + E_1)F(f^{y}, E_{y+1})(w) \\ &= (F(E_{x'-(f(y)+1)}, f_y) + {}^{(f_y)}F(x'-(f(y)+1)))(F(f^{y}, E_y) + {}^{y}F(f^{y}))(w) \\ &= (F(E_{x'-(f(y)+1}, f_y) + {}^{(f_y)}F(x'-(f(y)+1)))(F(x'-(f(y)+1), y) + {}^{y}F(f^{y})(u)) \\ &= F(x'-(f(y)+1), f(y)) + {}^{(f_y)}F(x'-(f(y)+1)^{y}F(f^{y})(u) \end{aligned}$$

hence $v = {}^{(f_y)}F(f^{y})(u) = h_{f(y)}(u)$.

(ii) This is proved similarily, and is left to the reader.

Proposition 3.6.3. (i) *If F is a dilator of kind Ω, if y is an ordinal, then $F \circ (\underline{y} + \mathrm{Id})$ is a dilator of kind Ω.*

(ii) *If F is a dilator of kind Ω, let $g \in I(y, y')$ and let $T = F \circ (\underline{g} + \mathrm{Id})$ be the natural transformation from $F \circ (\underline{y} + \mathrm{Id})$ to $F \circ (\underline{y}' + \mathrm{Id})$ defined by $T(x) = F(g + E_x)$ (so $T(f) = F(g+f)$). Then $T \in \Omega\tilde{I}(F \circ (\underline{y} + \mathrm{Id}), \tilde{F} \circ (\underline{y}' + \mathrm{Id}))$.*

(iii) *If F, G are dilators of kind Ω, if $U \in \Omega I(F, G)$, then $U \circ (\underline{y} + \mathrm{Id}) \in \Omega I(F \circ (\underline{y} + \mathrm{Id}), G \circ (\underline{y} + \mathrm{Id}))$.*

Proof. Left to the reader.

Definition 3.6.4. (i) If F is a dilator of kind Ω, if y is an ordinal, define the dilator $^*_yF$ by:

$$F \circ (\underset{\cdot}{\cdot} + \mathrm{Id}) = {}^*_yF + F' \quad \text{for some } F' \text{ perfect.}$$

(ii) If F is dilator of kind Ω, let $g \in I(y, y')$; one defines a natural transformation $^*_gF$ from $^*_yF$ to $^*_{y'}F$ by:

$$G \circ (\underset{\sim}{g} + \mathrm{Id}) = {}^*_gF + T' \quad \text{for some } T'.$$

(iii) If F, G are dilators of kind Ω, if $T \in \Omega I(F, G)$, if y is an ordinal, then define $^*_yT$, a natural transformation from $^*_yF$ to $^*_yG$:

$$T \circ (\underset{\sim}{y} + \mathrm{Id}) = {}^*_yT + T' \quad \text{for some } T'.$$

Definition 3.6.5. (i) If F is a dilator of kind Ω, if y is an ordinal. then define the dilator $_yF$ by:

$$_{y+1}^{\;\;*}F = {}^*_yF \circ (\underset{\sim}{1} + \mathrm{Id}) + {}_yF.$$

(ii) If F is a dilator of kind Ω, if $g \in I(y, y')$, define a natural transformation $_gF$ from $_yF$ to $_{y'}F$, by:

$$(g + E_1)^{*F} = {}^*_gF \circ (\underset{\sim}{1} + \mathrm{Id}) + {}_gF.$$

(iii) If F, G are dilators of kind Ω, if $T \in \Omega I(F, G)$, if y is an ordinal, then define a natural transformation $_yT$ from $_yF$ to $_yG$:

$$_{y+1}^{\;\;*}T = {}^*_yT \circ (\underset{\sim}{1} + \mathrm{Id}) + {}_yT.$$

Theorem 3.6.6. (i) *Suppose the F is a dilator of kind Ω; then* SEP(F)*is the bilator G defined by; ($f \in I(x, x')$, $g \in I(y, y')$)*

$$G(x, 0) = {}^*_0F(x), \qquad G(f, E_0) = {}^*_0F(f), \tag{1}$$

$$G(x, y+1) = G(x, y) + {}_yF(x), \qquad G(f, g + E_1) = G(f, g) + {}_gF(f), \tag{2}$$

$$G(f, g + E_{01}) = G(f, g) + E_{0_y, F(x')}, \tag{3}$$

$$G(x, \sup(y_i)) = \sup(G(x, y_i)), \qquad G(f, \cup g_i) = \bigcup G(f, g_i). \tag{4}$$

(ii) *Suppose that F and G are dilators of kind Ω, and that $T \in \Omega I(F, G)$, then* SEP(T) *is the natural transformation U from* SEP(F) *to* SEP(G) *defined by:*

$$U(x, 0) = {}^*_0T(x), \qquad U(x, y+1) = U(x, y) + {}_yT(x),$$

$$U(x, \sup(y_i)) = \bigcup U(x, y_i).$$

Proof. A preliminary step will be to investigate the meaning of Definitions 3.6.4 and 3.6.5. Assume that F is perfect $\neq \underset{\sim}{1}$, then $^*_yF(x)$ is exactly the set of all elements.

$z = (z_0; x_0, \ldots, x_{n-1}; y + x)_F$, such that $x_{\sigma_{z_0,n}(0)} < y$ (by Proposition 2.3.9, this set is an initial segment of $F(y + x)$, i.e., an ordinal. Let $p = \sigma_{z_0,n}(0)$. Assume that

$x_p < y$, and let $G = F \circ (y + \mathrm{Id})$:
–if x is limit, and $z \geqslant {}^*_y F(x)$, then the points $G(f)(z)$ are cofinal in $G(x)$ when f varies through $P(x, x)$; but

$$G(f)(z) = (z_0; x_0, \ldots, x_p, (E_y + f)(x_{p+1}), \ldots, (E_y + f)(x_{n-1}); y + x)_F,$$

hence

$$G(f)(z) \geqslant (z_0; x_0, \ldots, x_{p-1}, y, y+1, \ldots, y+n-1-p; y+x)$$

by Theorem 3.2.4, contradiction.
–in general, one takes the images under $f = E_{1_\omega} \cdot E_x$, and one applies the case x limit.

Conversely, assume that $z < {}^*_y F(x)$. We show that $x_p < y$: if one writes ${}^*_y F(x) = (a_0; b_0, \ldots, b_{m-1}; y + x)_F$, by what we have just proved, $b_q \geqslant y$, with $q = \sigma_{a_0,m}(0)$. If x is limit, then, for a well-chosen $f \in P(x, x)$, and if $x_p \geqslant y$, $G(f)(z)$ will be strictly greater than ${}^*_y F(x)$ (it suffices to have $y + f(x_p - y) > b_q$, and to apply Proposition 2.3.9); from this it follows that the points $G(g)(z)$ are cofinal in $G(x)$ when g varies through $P(x, x)$, a contradiction with the choice of z. If x is not limit, one proceeds as above.

If F is of kind Ω, then $F = {}^*_0 F + F_1$, with F_1 perfect, and one sees easily ${}_y F = {}_y F_1$, ${}_g F = {}_g F_1$. So, we study ${}_y F$ when F is perfect $\neq 1$.

If F is perfect, then ${}_y F(x)$ is the order type of the set (in fact an interval) $X_{x,y}$ of all elements $z = (z_0; x_0, \ldots, x_{n-1}; y + 1 + x)$, such that $x_p = y$, with $p = \sigma_{z_0,n}(0)$. This follows immediately from the characterization above of ${}^*_y F(x)$. If $f \in I(x, x')$, ${}_y F(f)(t) = u$ iff the uth element of $X_{x,y}$ is sent by $F(E_{y+1} + f)$ on the tth element of $X_{x',y}$. If $g \in I(y, y')$, then ${}_g F(x)(u) = t$ iff the uth element of $X_{x,y}$ is sent by $F(g + E_{1+x})$ on the tth element of $X_{x,y'}$.

Now, it is possible to prove Theorem 3.6.6.

(i) By definition, if $F = {}^*_0 F + F_1$, with F_1 perfect, then

$$G(x, y) = {}^*_0 F(x) + F_1(x, y), \qquad G(f, g) = {}^*_0(f) + F_1(f, g).$$

(1) $F_1(x, 0) = 0$ by the fact that $\overline{F(x, 0)} = \emptyset$, hence $F_1(f, E_0) = E_0$. So

$$G(x, 0) = {}^*_0 F(x), \qquad G(f, E_0) = {}^*_0 F(f)$$

(2) If $t \in \overline{F_1(x, y)}$, then $F_1(E_{yy+1} + E_x)(t) \in \overline{F_1(x, y+1)}$; and, if $z \in \overline{F_1(x, y+1)}$, but $z \notin \mathrm{rg}(F_1(E_{yy+1} + E_x))$, then $z = (z_0; x_0, \ldots, x_{n-1}; y + 1 + x)_{F_1}$, and $x_p = y$, with $p = \sigma_{z_0,n}(0)$. Hence ($z \in \overline{F_1(x, y+1)}$ and $z \in \mathrm{rg}(F_1(E_{yy+1} + E_x))$) iff $z \in X_{x,y}$; and for all z and t, $z \in X_{x,y}$ and $t \in \overline{F_1(x, y+1)} \cap \mathrm{rg}(F_1(E_{yy+1} + E_x))$ implies that $z > t$. One deduces that $F_1(x, y+1) = F_1(x, y) + {}_y F_1(x)$, hence,

$$F(x, y+1) = {}^*_0 F(x) + F_1(x, y+1) = F(x, y) + {}_y F_1(x) = F(x, y) + {}_y F(x).$$

Let $v = F_1(x, y) + u$, with $v < F_1(x, y+1)$, and $f \in I(x, x')$. $F_1(f, E_{y+1})(v) = F_1(x', y) + t$, iff the uth point of $X_{x,y}$ is sent by $F_1(E_{y+1} + f)$ on the tth point of $X_{x',y}$. Hence

$$F_1(f, E_{y+1})(F_1(x, y) + u) = F_1(x', y) + t \quad \text{iff } {}_y F_1(f)(u) = t.$$

This shows that

$$F_1(f, E_{y+1}) = F_1(f, E_y) + {}_yF_1(f).$$

Let $v = F_1(x, y) + u$, with $v < F_1(x, y+1)$, and $g \in I(y, y')$; $F_1(E_x, g + E_1)(v) = F_1(x, y') + t$ iff the uth point of $X_{x,y}$ is sent by $F_1(g + E_1 + E_x)$ on the tth point of $X_{x,y'}$. Hence

$$F_1(E_x, g + E_1)(F_1(x, y) + u) = F_1(x, y') + t \quad \text{iff } {}_gF_1(x) = t.$$

This shows that

$$F_1(E_x, g + E_1) = F_1(E_x, g) + {}_gF_1(x).$$

Finally

$$F_1(f, g + E_1) = F_1(E_{x'}, g + E_1)F_1(f, E_{y+1}) = F_1(f, g) + {}_gF_1(f),$$

and

$$F(f, g + E_1) = {}^*_0F(f) + F_1(f, g + E_1) = F(f, g) + {}_gF(f).$$

(3) and (4) are general properties of bilators.

(ii) This is straightforward and left to the reader.

Remark 3.6.7. If F is of kind Ω, then the bilator $\mathrm{SEP}(F)^y$ of Theorem 3.5.1 is equal to ${}^*_0F + \sum_{y'<y} {}_{y'}F$.

Remark 3.6.8. If F is of kind Ω, then $H(x, y) = {}_yF(x)$, $H(f, g) = {}_gF(f)$ defines a functor from ON×ON to ON commuting to lim and &, which is not at all a bilator in general. In general, if H is such a two variable functor, it is possible to define (provided H is non zero) a bilator $G = \int H(\cdot, y)\, dy$, by:

$$G(x, y) = \sum_{u<y} G(x, u), \qquad G(f, g) = \sum_{u<y'} h_u, \quad \text{when } g \in I(y, y'),$$

and $h_u = E_{OG(x',u)}$ if $u \notin \mathrm{rg}(g)$, $h_u = G(f, g_u)$ if $u \in \mathrm{rg}(g)$, and $g_u \in I(g^{-1}(u), u)$ is defined by $g_u(t) = g(t)$. This definition is exactly similar to Example 2.4.9(i).

Conversely, if G is a bilator, then one defines a functor $(d/dy)\ G = H$ by

$$G(x, y) + H(x, y) = G(x, y+1), \qquad G(f, g) + H(f, g) = G(f, g + E_1).$$

The following equalities are immediate, for the specific H defined by $H(x, y) = {}_yF(x)$, $H(f, g) = {}_gF(f)$:

$$\mathrm{SEP}(F) = {}^*_0F + \int H(\cdot, y)\, dy,$$

$$H = \frac{d}{dy}\,\mathrm{SEP}(F)$$

(the notations are not very good...).

4. The category DIL of dilators

4.1. The category DIL

Definition 4.1.1. The following data define a category DIL:
objects: dilators,
morphisms from F to G: the set $I(F, G)$ of natural transformations from F to G.

Definition 4.1.2. If F and G are functors from category $\mathscr{C}$ to category $\mathscr{D}$, if T is a natural transformation from F to G, if f is a $\mathscr{C}$-morphism from x to y, then one defines a $\mathscr{D}$-morphism $T(f)$ from $F(x)$ to $G(y)$ by:

$$T(f) = G(f)T(x) = T(y)F(f).$$

Theorem 4.1.3. (*We recall that the composition of natural transformations is defined by:* $(TU)(x) = T(x)U(x)$. *TU is the composition of the morphisms T and U in the category DIL.*)

Let (F_i, T_{ij}) *be a direct system in* DIL, *then*

(i) *If for all* $x \in \mathrm{ON}$, *the direct limit* $\varinjlim(F_i(x), T_{ij}(x))$ *exists, then* (F_i, T_{ij}) *has a direct limit in* DIL.

(ii) *Conversely, assume that* $(F, T_i) = \varinjlim_I(F_i, T_{ij})$ *and that* $(x, f_l) = \varinjlim_L(x_l, f_{lm})$, *then*

$$(F(x), T_i(f_l)) = \varinjlim_{I \times L}(F_i(x_l), T_{ij}(f_{lm})).$$

Proof. (i) Let $(F(x), T_i(x)) = \varinjlim(F_i(x), T_{ij}(x))$, and if $f \in I(x, y)$, define $F(f) = \varinjlim(F_i(f))$. F is clearly a functor from ON to ON, so it remains to prove commutation to $\varinjlim$ and to &:

F commutes to $\varinjlim$: if $(x, f_l) = \varinjlim_L(x_l, f_{lm})$, then, using Corollary 1.4.6:

$$\begin{aligned}
&\varinjlim_I(F(x_l), F(f_{lm}))\\
&\quad= \varinjlim_L\Big(\varinjlim_I{}^*(F_i(x_l), T_{ij}(x_l)), \varinjlim_I(F_i(f_{lm}))\Big)\\
&\quad= \Big(\varinjlim_I{}^*\Big(\varinjlim_L{}^*(F_i(x_l), F_i(f_{lm})), \varinjlim_L(T_{ij}(x_l))\Big), \varinjlim_I\Big(\varinjlim_m(F_i(f_{lm}))\Big)\Big)\\
&\quad= \Big(\varinjlim_I{}^*(F_i(x), T_{ij}(x)), \varinjlim_I(F_i(f_l))\Big)\\
&\quad= (F(x), F(f_l)).
\end{aligned}$$

(One applies Corollary 1.4.6 with $z_{il} = F_i(x_l)$, $h_{il,jm} = T_{ij}(f_{lm})$ so $h_{il,im} = F_i(f_{lm})$ and $h_{il,jl} = T_{ij}(x_l)$).

F commutes to &: one uses Theorem 1.5.6:

$$\begin{aligned}
F(f \,\&\, g) &= \varinjlim(F_i(f \,\&\, g)) = \varinjlim(F_i(f) \,\&\, F_i(g))\\
&= \varinjlim(F_i(f)) \,\&\, \varinjlim(F_i(g)) = F(f) \,\&\, F(g).
\end{aligned}$$

F is therefore a dilator, and (F, T_i) enjoys obviously the conditions 1.3.3(i)–(iii). We verify (iv): if (G, U_i) is another solution of (i)–(iii), then for all x, $(G(x), U_i(x))$ enjoys (i)–(iii) w.r.t. the direct system $(F_i(x), T_{ij}(x))$, hence there exists an unique morphism $V(x)$ from $F(x)$ to $G(x)$ such that $U_i(x) = V(x)T_i(x)$. The family $V = (V(x))$ is obviously a natural transformation from F to G such that $U_i = VT_i$ for all i, and we have seen that V is unique with this property.

(ii) $(F(x), T_i(x))$ enjoys conditions (i)–(iii) of direct limits, hence (by Corollary 1.4.4) the direct limit of this system exists, so by (i) above $(F(x), T_i(x)) = \varinjlim(F_i(x), T_{ij}(x))$, and by Theorem 1.4.5(ii)

$$\begin{aligned}
&\varinjlim_{I\times L}(F_i(x_l), T_{ij}(f_{lm}))\\
&\quad = \Big(\varinjlim_I{}^{*}\Big(\varinjlim_L{}^{*}(F_i(x_l), F_i(f_{lm})), \varinjlim_L(T_{ij}(x_l))\Big), \varinjlim_j\Big(\varinjlim_m(T_{ij}(f_{lm}))\Big)\Big)\\
&\quad = \Big(\varinjlim_I{}^{*}(F_i(x), T_{ij}(x)), \varinjlim_j(T_{ij}(f_l))\Big)\\
&\quad = (F(x), T_i(f_l)).
\end{aligned}$$

Remarks 4.1.4. (i) in Theorem 4.1.3(ii), one can take the following specific cases:
$-(x_l, f_{lm}) = (x, E_x)$, then $(F(x), T_i(x)) = \varinjlim(F_i(x), T_{ij}(x))$;
$-I = L$; since the pairs (i, i) form (because I is directed) a cofinal subset of $I \times I$, we get: $(F(x), T_i(f_i)) = \varinjlim(F_i(f), T_{ij}(f_{ij}))$.

(ii) If $(F(n), T_i(n)) = \varinjlim(F_i(n), T_{ij}(n))$ for all n, then $(F, T_i) = \varinjlim(F_i, T_{ij})$. The hypothesis implies that F and the direct limit G of (F_i, T_{ij}) coincide on the category $\mathrm{ON}{<}\omega$, so they are equal.

(iii) The direct limit, when it exists, is unique.

(iv) If (F, T_i) enjoys properties (i)–(iii) of direct limits, then there is a direct limit for (F_i, T_{ij}).

Example 4.1.5. (i) Suppose that $(F^z)_{z<x}$ is a family of dilators, then we shall construct the sum $\sum_{z<x} F^z$ as a direct limit in DIL. Take a direct system (x_i, f_{ij}) with direct limit (x, f_i), and define, when $i \in I$, $F^{z,i}$ by: $F^{z,i} = F^{f_i(z)}$, and $T^{z,i}$ to be the identity of $F^{z,i}$. If $i < j$, then define $T^{z,ij} = T^{z,i}$. Observe that $T^{z,i} \in I(F^{z,i}, F^{f_i(z)})$ and that $T^{z,ij} \in I(F^{z,i}, F^{f_{ij}(z),j})$. Define

$$F = \sum_{z<x} F^z, \qquad F_i = \sum_{z<x_i} F^{z,i},$$

$$T_i = \sum_{z<f_i} T^{z,i}, \quad T_{ij} = \sum_{z<f_{ij}} T^{z,ij}$$

(we are summing the families $(T^{z,i})$ of natural transformations from $F^{z,i}$ to $F^{f_i(z)}$ and $(T^{z,ij})$ of natural transformations from $F^{z,i}$ to $F^{f_{ij}(z),j}$, according to Definition 3.1.1(ii)). It is immediate that $(F, T_i) = \varinjlim(F_i, T_{ij})$.

(ii) If $(G^z)_{z<y}$ is another family of dilators, if $(y, g_l) = \varinjlim_L(y_l, g_{lm})$, then it is possible, as in (i) to express $G = \sum_{z<y} G^z$ as a direct limit. $(G, U_l) = \varinjlim_L (G_l, U_{lm})$. Suppose that $h \in I(x, y)$ is such that $h = \varinjlim(h_i)$, where (h_i) is a direct system of morphisms between (x_i, f_{ij}) and (y_l, g_{lm}), with associated function φ. Suppose that for $z<x$, $V^z \in I(F^z, G^{f(z)})$, then we shall express V as $\varinjlim(V_i)$, for a certain system (V_i) between (F_i, T_{ij}) and (G_l, U_{lm}), with associated function φ. Let $V^{i,z}$ be $V^{h_i(z)}$, then $V^{i,z} \in I(F^{i,z}, G^{\varphi(i), h_i(z)})$, and so one can define $V_i \in I(F_i, G_{\varphi(i)})$ by $V_i = \sum_{z<h_i} V^{i,z}$. It is immediate that (V_i) is a direct system of morphisms between (F_i, T_{ij}) and (G_l, U_{lm}) and that $V = \varinjlim(V_i)$.

(iii) In (i), if one takes $I = x$, $x_i = i$, $f_{ij} = E_{ij}$, then $(F, T_i) = \varinjlim(F_i, T_{ij})$, with $F_i = \sum_{j<i} F^j$, $T_i(x)(z) = z$, $T_{ij}(x)(z) = z$ for all z.

(iv) In (ii), if one takes $L = y$, $y_l = l$, $g_{lm} = E_{lm}$, if (x_i, f_{ij}) is as in (iii), then with $\varphi = h$, we have $V = \varinjlim(V_i)$, with $V_i = \sum_{j<i} V^j$.

(v) In order to verify (i)–(iv) (essentially (i), (ii)), the simplest would be to wait for the simple criterion Proposition 4.2.6(iv)'.

Remark 4.1.6. Most of properties of direct limits in ON still hold in DIL, for instance, Theorems 1.4.3, 1.4.5 and Corollary 1.4.4. The reason for this is that Remark 4.1.4(iv) is the only thing needed in order to carry the proofs. These properties will also hold for the category PIL which parallels OL.

4.2. *Morphisms in* DIL

Notation 4.2.1. E_F will denote the identity of F. However, since $E_F(x) = E_{F(x)}$, $E_F(f) = F(f)$, we shall usually abbreviate E_F in F.

Definition 4.2.2. (i) If F is a dilator, define $\mathrm{rg}(F)$ by:

$$\mathrm{rg}(F) = \{(z; n); z = (z; 0, \ldots, n-1; n)\}$$

(the notations being taken with respect to F).

(ii) If F, G are dilators, if $T \in I(F, G)$, then define $\mathrm{rg}(T)$:

$$\mathrm{rg}(T) = \{(z; n); (z; n) \in \mathrm{rg}(G) \text{ and } (z; 0, \ldots, n-1; n) \in \mathrm{rg}(T(n))\}$$

(the notations are taken with respect to G, of course).

Remark 4.2.3. $\mathrm{rg}(F)$ (Definition 4.2.2(i)) is equal to $\mathrm{rg}(E_F) = \mathrm{rg}(F)$ (Definition 4.2.2(ii), when F is an abbreviation for E_F).

Theorem 4.2.4. *If F, G are dilators, if $T \in I(F, G)$, then (the notations being taken w.r.t. G):*

(i) *For all $x \in \mathrm{ON}$, $\mathrm{rg}(T(x))$ is the set of ordinals that can be written $(z; x_0, \ldots, x_{n-1}; x)$ for some $(z; n) \in \mathrm{rg}(T)$ and $x_0, \ldots, x_{n-1} < x$.*

(ii) *If $f \in I(x, y)$, then $\mathrm{rg}(T(f))$ is the set of all ordinals that can be written $(z; y_0, \ldots, y_{n-1}; y)$ for some $(z; n) \in \mathrm{rg}(T)$ and $y_0, \ldots, y_{n-1} \in \mathrm{rg}(f)$.*

Proof. (i) If $a=(z;x_0,\dots,x_{n-1};x)$, then:
–if $a\in\mathrm{rg}(T(x))$, define $f\in I(n,x)$ by $f(i)=x_i$. Using Proposition 2.3.15, find $b\in F(n)$ such that $a=T(x)F(f)(b)$. But also $a=G(f)T(n)(b)$, and so $T(n)(b)=(z;0,\dots,n-1;n)$ (by Remark 2.3.13(iii)). So $(z;n)\in\mathrm{rg}(T)$.
–conversely, if $(z;n)\in\mathrm{rg}(T)$, then $a=G(f)T(n)(b)=T(x)F(f)(b)$ for some b by hypothesis, hence $a\in\mathrm{rg}(T(x))$.

(ii) This is immediate from the remark that $\mathrm{rg}(T(f))=G(f)(\mathrm{rg}(T(x)))$.

Theorem 4.2.5. *Let G be a dilator, and let $X\subset\mathrm{rg}(G)$, then there exists an unique dilator F and an unique $T\in I(F,G)$ such that* $\mathrm{rg}(T)=X$.

Proof. $\mathrm{rg}(T(x))$ is uniquely determined by Theorem 4.2.4(i). Define $F(x)$ by $T(x)\in I(F(x),G(x))$, and, when $f\in I(x,y)$, define $F(f)$ by $T(y)F(f)=G(f)T(x)$. It is immediate that F is a functor from ON to ON, that T is a natural transformation from F to G, and that this is the only possible solution of the problem. It remains to show that F is a dilator:

F commutes to $\varinjlim$: if $(x,f_i)=\varinjlim(x_i,f_{ij})$, then let $z\in F(x)$. If $T(x)(z)=(a;x_0,\dots,x_{n-1};x)$, then $(a;n)\in X$ by definition of T, and there exists $i\in I$ such that that

$$T(x)(z)=G(f_i)(a;f_i^{-1}(x_0),\dots,f_i^{-1}(x_{n-1});x_i).$$

Let $u_i=(a;f_i^{-1}(x_0),\dots,f_i^{-1}(x_{n-1});x_i)$, then $u_i\in\mathrm{rg}(T(x_i))$. So $u_i=T(x_i)(v_i)$; we conclude that $T(x)(z)=G(f_i)T(x_i)(v_i)=T(x)F(f_i)(v_i)$. So $z=F(f_i)(v_i)$. By Example 1.3.6(iv)′, $(F(x),F(f_i))=\varinjlim(F(x_i),F(f_{ij}))$.

F commutes to &: if $f_i\in I(x_i,y)$ $(i=1,2,3)$, if $f_3=f_1\,\&\,f_2$, then

$$\begin{aligned}\mathrm{rg}(F(f_3))&=T(y)^{-1}(\mathrm{rg}(G(f_3)))=T(y)^{-1}(\mathrm{rg}(G(f_1))\cap\mathrm{rg}(G(f_2)))\\&=T(y)^{-1}(\mathrm{rg}(G(f_1)))\cap T(y)^{-1}(\mathrm{rg}(G(f_2)))\\&=\mathrm{rg}(F(f_1))\cap\mathrm{rg}(F(f_2)).\end{aligned}$$

By Theorem 1.5.5 F commutes to pull-backs.

Proposition 4.2.6. $(F,T_i)=\varinjlim(F_i,T_{ij})$ *is equivalent to the conditions* 1.3.3(i)–(iii) *and to condition* (iv)′:

(iv)′ $\mathrm{rg}(F)=\bigcup_{i\in I}\mathrm{rg}(T_i)$.

Proof. If $(F,T_i)=\varinjlim(F_i,T_{ij})$, then $(F(n),T_i(n))=\varinjlim(F_i(n),T_{ij}(n))$. So, if $(z;n)\in\mathrm{rg}(F)$, then $(z;0,\dots,n-1;n)\in\mathrm{rg}(T_i(n)$ for some i by Example 1.3.6. so $(z;n)\in\mathrm{rg}(T_i)$. Hence (iv)′ holds.

Conversely assume (iv)′, then we show that $(F(x),T_i(x))=\varinjlim(F_i(x),T_{ij}(x))$ for all $x\in\mathrm{ON}$. If $(z;x_0,\dots,x_{n-1};x)\in F(x)$, then choose i such that $(z;n)\in\mathrm{rg}(T_i)$, then $(z;x_0,\dots,x_{n-1};x)\in\mathrm{rg}(T_i(x))$ by Theorem 4.2.4(i). So the result follows by Example 1.3.6. Applying now the construction of Remark 4.1.4(i), one gets $(F,T_i)=\varinjlim(F_i,T_{ij})$.

Theorem 4.2.7. *If* $T_1 \in I(F_1, G)$, *if* $T_2 \in I(F_2, G)$, *then* $T_1 \,\&\, T_2$ *exists and is uniquely determined by* $\mathrm{rg}(T_1 \,\&\, T_2) = \mathrm{rg}(T_1) \cap \mathrm{rg}(T_2)$.

Proof. Definining F_3 and $T_3 \in I(F_3, G)$ by $\mathrm{rg}(T_3) = \mathrm{rg}(T_1) \cap \mathrm{rg}(T_2)$ and $T_{13} \in I(F_3, F_1)$, $T_{23} \in I(F_3, F_2)$ by $T_3 = T_1 T_{13} = T_2 T_{23}$, then Definition 1.5.1(i) is satisfied. Assume that G_3, U_3, U_{13}, U_{23} is another solution of Definition 1.5.1(i). Since $\mathrm{rg}(U_3) \subset \mathrm{rg}(T_3)$, it is possible to define $V \in I(G_3, F_3)$ by: $U_3 = T_3 V$, then $U_{13} = T_{13} V$ and $U_{23} = T_{23} V$, and V is uniquely determined. From this we get Definition 1.5.1(ii), i.e., $T_3 = T_1 \,\&\, T_2$.

Theorem 4.2.8. *Assume that* $T_i \in I(F_i, G)$ $(i = 1, 2, 3)$, *then the following are equivalent:*

(i) $T_3 = T_1 \,\&\, T_2$.
(ii) For all f_1, f_2, f_3 *such that* $f_3 = f_1 \,\&\, f_2$, *then* $T_3(f_3) = T_1(f_1) \,\&\, T_2(f_2)$.
(iii) *For all* $x \in \mathrm{ON}$, $T_3(x) = T_1(x) \,\&\, T_2(x)$.
(iv) *For all* n, $T_3(n) = T_1(n) \,\&\, T_2(n)$.

Proof. (i) $\to$ (ii). $(z; y_0, \ldots, y_{n-1}; y) \in \mathrm{rg}(T_i(f_i))$ iff $(z; n) \in \mathrm{rg}(T_i)$ and $y_0, \ldots, y_{n-1} \in \mathrm{rg}(f_i)$. Using $\mathrm{rg}(T_1) \cap \mathrm{rg}(T_2) = \mathrm{rg}(T_3)$ and $\mathrm{rg}(f_1) \cap \mathrm{rg}(f_2) = \mathrm{rg}(f_3)$, one concludes that

$$\mathrm{rg}(T_1(f_1)) \cap \mathrm{rg}(T_2(f_2)) = \mathrm{rg}(T_3(f_3)).$$

(ii) $\to$ (iii). Take $f_1 = f_2 = f_3 = E_x$.

(iii) $\to$ (iv). Trivial.

(iv) $\to$ (i). If $T_3(n) = T_1(n) \,\&\, T_2(n)$, then $(z; n) \in \mathrm{rg}(T_3)$ iff it belongs to $\mathrm{rg}(T_1)$ and $\mathrm{rg}(T_2)$. So $\mathrm{rg}(T_3) = \mathrm{rg}(T_1) \cap \mathrm{rg}(T_2)$, and apply Theorem 4.2.7.

Corollary 4.2.9. *If* $T \in I(F, G)$, *if* $f \in I(x, y)$, *then* $T(f) = T(y) \,\&\, G(f)$.

Proof. $T \,\&\, G = T$ because $\mathrm{rg}(T) \subset \mathrm{rg}(G)$. Also $E_y \,\&\, f = f$ by Theorem 4.2.8(ii) $T(f) = T(E_y) \,\&\, G(f) = T(y) \,\&\, G(f)$.

Definition 4.2.10. If $T, U \in I(F, G)$, then $T \leqslant U$ means that

$$\forall\, x \in \mathrm{ON} \quad T(x) \leqslant U(x) \quad \text{(see Definition 2.3.8)}.$$

Proposition 4.2.11. *If* $T, U \in I(F, G)$, *then the following are equivalent:*

(i) $T \leqslant U$.
(ii) For all n, $T(n) \leqslant U(n)$.
(iii) For all x, y, f, g, with x, y ordinals, $f, g \in I(x, y)$, $f \leqslant g \to T(f) \leqslant U(g)$.

Proof. (i) $\to$ (iii). If $T \leqslant U$, then $T(x) \leqslant U(x)$. By Proposition 2.3.10, $G(f) \leqslant G(g)$. Hence

$$T(f) = G(f)T(x) \leqslant G(f)U(x) \leqslant G(g)U(x) = U(g).$$

(iii) → (ii). Take $f = g = E_n$.

(ii) → (i). If $T(n) \leqslant U(n)$ for all n, let $z \in F(x)$. We show that $T(x)(z) \leqslant U(x)(z)$. Write $z = F(f)(z_0)$, with $f \in I(n, x)$, then, by hypothesis $T(n)(z_0) \leqslant U(n)(z_0)$, so

$$T(x)(z) = G(f)T(n)(z_0) \leqslant G(f)U(n)(z_0) = U(x)(z).$$

Proposition 4.2.12. *If $T, U \in I(F, G)$, then $T \leqslant U$ iff there exists a dilator H and $V \in I(G, H)$, $W \in I(H, H)$ such that $VU = WVT$.*

Proof. (The difficulty of the proof is in no relation to the interest of the proposition, so the reader is advised to omit to read it, unless he is especially interested in this property.) Observe first that the condition is obviously sufficient. If $VU = WVT$, then $VU(x) = WVT(x)$, so $V(x)U(x) = W(x)V(x)T(x)$ for all x, and from $W(x)(z) \geqslant z$ for all z, one gets $V(x)U(x) \geqslant V(x)T(x)$, and this forces $U(x) \geqslant T(x)$. The necessity of the condition is proved by means of a series of lemmas.

Lemma 4.2.13. *Take the set $G(x) \times \omega$ partially ordered by $(a, n) \leqslant (b, m)$ iff $n = m$ and $a \leqslant b$, and identify the points $(U(x)(z), n)$ with $(T(x)(z), n+1)$. The resulting set S_x, partially ordered by R_x, is well founded.*

Proof. The basic remark is that the identifications induce no identification of elements (a, n) and (b, n) when $a \neq b$, i.e., if one views R_x as a preorder on $G(x) \times \omega$, its restriction to subsets $G(x) \times \{n\}$ is the order $G(x)$. Given a strictly decreasing sequence for R_x, then one can construct a sequence (a_n, p_n) in $G(x) \times \omega$, such that for all n:

(i) either $p_{n+1} = p_n$ and $a_{n+1} < a_n$,

(ii) or $p_{n+1} = p_n + 1$, $a_n = U(x)(z)$, $a_{n+1} = T(x)(z)$ for some z,

(iii) or $p_n = p_{n+1} + 1$, $a_n = T(x)(z)$, $a_{n+1} = U(x)(z)$ for some z, and case (i) occurs infinitely many often.

Subcase 1. The values p_n are bounded, then p_n takes the value p for infinitely many n, and for n in an infinite set X, $(a_n, p_n) = (a_n, p)$ is a strictly decreasing sequence in $G(x) \times \{p\}$, contradiction.

Subcase 2. The values p_n are unbounded, then it is easy to construct a subsequence such that case (iii) never occurs (because a case (iii) is always 'cancelled' by a case (ii)). But, since $T \leqslant U$, one gets $a_{n+1} \leqslant a_n$ when case (ii) holds, so the full subsequence (b_n, q_n) is such that $b_{n+1} \leqslant b_n$ for all n, with infinitely many values (corresponding to case (i)) such that $b_{n+1} < b_n$: this contradicts the fact that $G(x)$ is an ordinal.

Lemma 4.2.14. *Replace R_x by R'_x: $(a, n)R'_x(b, m)$ iff for some $y \in \mathrm{ON}$ and $f \in I(x, y)$, $(G(f)(a), n)R_y(G(f)(b), m)$, then R'_x (which is an extension of R_x) is well founded.*

Proof. We shall prove the existence of a strictly increasing function from R'_x to $R_{\omega(1+x)}$. This establishes well-foundedness of R'_x. The function is defined by $k((a, n)) = (G(E_{0\omega} + E_{1\omega}E_x)(a), n)$, i.e.,

$$k((z_0; a_0, \ldots, a_{p-1}; x)_G, n)$$
$$= ((z_0; \omega(1+a_0), \ldots, \omega(1+a_{p-1}); \omega(1+x))_G, n).$$

Assume $A = (G(f)(a), n)R_y(G(f)(b), m) = B$. So there is a finite sequence $A = (c_0, p_0), \ldots, B = (c_i, p_i)$, such that for all $j < i$, one of (i)–(iii) of Lemma 4.2.13 holds. Let X be the (finite) set of all coefficients occuring in the normal forms of the points c_j. So X is a finite subset of y, and there exists a strictly increasing function g from X to $\omega(1+x)$, such that $g(f(t)) = \omega(1+t)$ for all t with $f(t) \in X$. If $c_j = (z_0; a_0, \ldots, a_{r-1}; y)$, define

$$d_j = (z_0; g(a_0), \ldots, g(a_{r-1}); \omega(1+x)).$$

It is immediate that $d_0 = a$, $d_i = b$, and the sequence d_j shows that $(a, n)R_{\omega(1+x)}(b, m)$. We have just shown that k is strictly increasing.

Lemma 4.2.15. *Extend R'_x into R''_x by deciding that, whenever (a, n) and (b, m) are uncomparable by means of R'_x, then $(a, n)R''_x(b, m)$ iff $n < m$. Then R''_x is a well-order.*

Proof. First consider the relation R^*_x defined on S_x, by: $(a, n)R^*_x(b, m)$ iff $(a, n)R_x(b, m)$ or (a, n), (b, m) uncomparable by means of R_x, and $n < m$. We show that R^*_x is a linear order. The only non trivial point is transitivity. Let us denote by $(a, n)U(b, m)$ the relation (a, n), (b, m) uncomparable modulo R_x and $n < m$, then we verify the following:

(i) $(a, n)U(b, m)$ and $(b, m)U(c, p)$ imply $(a, n)R^*_x(c, p)$; simply observe that, if $(c, p)R_x(a, n)$, then one can find b' such that $(c, p)R^*_x(b', m)$ and $(b', m)R^*_x(a, n)$. If $(b', m)R_x(b, m)$, we obtain a contradiction with $(b, m)U(c, p)$, and if $(b, m)R_x(b', m)$, we contradict $(a, n)U(b, m)$. So $(c, p)R_x(a, n)$ is impossible, so either $(a, n)R_x(c, p)$, or $(a, n)U(c, p)$.

(ii) $(a, n)U(b, m)$ and $(b, m)R_x(c, m)$ imply $(a, n)R^*_x(c, m)$. Observe that $(c, m)R_x(a, n)$ is impossible (this contradicts $(a, n)U(b, m)$).

(iii) $(a, n)R_x(b, n)$ and $(b, n)U(c, m)$ imply $(a, n)R^*_x(c, m)$: as in (ii).

(iv) $(a, n) = (b, n+1)$ and $(b, n+1)U(c, m)$ imply $(a, n)R^*_x(c, m)$: trivial.

(v) $(a, n)U(b, m)$ and $(b, m) = (c, m+1)$ imply $(a, n)R^*_x(c, m+1)$: trivial.

(vi) $(a, n+1) = (b, n)$ and $(b, n)U(c, m)$ imply $(a, n+1)R^*_x(c, m)$: trivial, except if $m = n+1$. But in that case, if $(c, m)R_x(a, m)$, one gets $(c, m)R_x(b, n)$, a contradiction.

(vii) $(a, n)U(b, m+1)$ and $(b, m+1) = (c, m)$ imply $(a, n)R^*_x(c, m)$: as in (vi).

Using (i)–(vii), transitivity is easily shown. Given a strictly decreasing sequence (a_n, p_n) in S_x for R^*_x, then one can extract (using, for instance Ramsey's theorem) a subsequence (b_n, q_n) such that either $(b_n, q_n)R_x(b_{n+1}, q_{n+1})$ for all n, or

$(b_n, q_n)U(b_{n+1}, q_{n+1})$ for all n. The first possibility is destroyed by Lemma 4.2.13, and the second is absurd, because (q_n) would be a strictly decreasing sequence of integers.

Now, it is easy to end the proof of the lemma. Simply observe that the function k of Lemma 4 2.14 is strictly increasing function from S_x (with order R_x'') to S_y (with order R_y^*, $y = \omega(1+x)$). So R_x'' is a well-order. (If one wants to be completely rigorous, it is necessary to prove that R_x'' is a linear order, but it is clear that $(a, n)R_x''(b, m)$ iff $k((a, n)R_y^* k((b, m)), \ldots)$

Lemma 4.2.16. *The functor* $H(x) = R_x''$, $H(f)(a, n) = (G(f)(a), n)$ *is (isomorphic to) a dilator.*

Proof. Immediate. The non trivial point is that $H(f)$ is a strictly increasing mapping, but this comes from the replacement of R_x by R_x'.

End of the proof of Proposition 4.2.12: define $V \in I(G, H)$ by $V(x)(z) = (z, 0)$, $W \in I(H, H)$ by $W(x)(z, n) = (z, n+1)$. then

$$\begin{aligned} V(x)U(x)(z) &= (U(x)(z), 0) = (T(x)(z), 1) \\ &= W(x)(T(x)(z), 0) = W(x)V(x)T(x)(z), \end{aligned}$$

hence $VU = WVT$.

Corollary 4.2.17. (i) *Let Φ be a functor from* DIL *to* DIL, *then*

$$T \leq U \rightarrow \Phi(T) \leq \Phi(U).$$

(ii) *Let Ψ be a functor from* DIL *to* ON, *then*

$$T \leq U \rightarrow \Psi(T) \leq \Psi(U).$$

(iii) *Let Θ be a functor from* ON *to* DIL, *then*

$$f \leq g \rightarrow \Theta(f) \leq \Theta(g).$$

Proof. Immediate application of Propositions 4.2.12 and of 2.3.9.

Remark 4.2.18. The very long and boring proof of Proposition 4 2.12 has at least one advantage: it can be easily generalized to functors of more complex types, without any essential change in the proof.

4.3. Strongly finite dilators

Definition 4.3.1. A dilator F is *strongly finite* iff there are only finitely many dilators G and morphisms $T \in I(G, F)$.

Theorem 4.3.2. *F is strongly finite iff* $\mathrm{rg}(F)$ *is finite.*

Proof. Obvious, since by Theorem 4.2.5, the natural transformations with target F are isomorphic with the subsets of $\mathrm{rg}(F)$. Precisely, if F is strongly finite, there are $2^{\mathrm{rg}(F)}$ distinct morphisms with target F.

Proposition 4.3.3. *F is strongly finite iff F is weakly finite and $F(n)$ is a polynomial of n.*

Proof. –If F is strongly finite, then

$$F(n) = a_0 + a_1 n + a_2 n(n-1)/2 + \cdots + a_k n(n-1)\cdots(n-k+1)/k!,$$

with

$$k = \sup\{i; \exists z\ (z, i) \in \mathrm{rg}(F)\}, \qquad a_i = \mathrm{card}(\{z; (z, i) \in \mathrm{rg}(F)\})$$

(see Remark 3.2.7(i))

–if F is weakly finite, then it is still possible to write

$$F(n) = a_0 + a_1 n + \cdots + a_k n(n-1)\cdots(n-k+1)/k! + \cdots,$$

with $a_i = \mathrm{card}(\{z; (z, i) \in \mathrm{rg}(F)\})$, and this infinite sum is a polynomial iff almost all coefficients a_k are zero.

Proposition 4.3.4. *A strongly finite dilator is primitive recursive.*

Proof. $F(k)$ is the set of all notations $(z; i_0, \ldots, i_{n-1}; k)$, when $(z; n) \in \mathrm{rg}(F)$ and $i_0 < \cdots < i_{n-1} < k$, and we have, when $f \in I(k, k')$

$$F(f)(z; i_0, \ldots, i_{n-1}; k) = (z; f(i_0), \ldots, f(i_{n-1}); k'),$$

so all we need to know is that the ordering of $F(k)$ is a recursive primitive function of k. But in order to compare $(z; i_0, \ldots, i_{n-1}; k)$ and $(z'; j_0, \ldots, j_{m-1}; k)$: it is sufficient by Proposition 2.3.17 to compare $(z; i'_0, \ldots, i'_{n-1}; l)$ and $(z'; j'_0, \ldots, j'_{m-1}; l)$ where the sequences i', j' satisfy $i'_r < j'_s$ iff $i_r < j_s$, and $i'_{n-1}, j'_{m-1} < n + m \leq l$. If l is equal to two times the degree of the polynomial $F(n)$, then it follows that the ordering of $F(l)$ determines completely (in a primitive recursive way) the ordering of $F(k)$ for all k.

Definition 4.3.5. The *degree* of the strongly finite dilator F is the degree of the polynomial $F(n)$.

Definition 4.3.6. A *k-thing* is a functor from $\mathrm{ON}{\leq}3k$ into $\mathrm{ON}{\leq}\omega$ such that:

(i) F commutes to pull-backs;

(ii) $\forall i \leq 3k\ \forall z < F(i)\ \exists j < k\ \exists f \in I(j, i)\ (z \in \mathrm{rg}(F(f)))$;

(iii) for all i, j, f, g, $i, j \leq 3k$, $f, g \in I(i, j)$, then $f \leq g \to F(f) \leq F(g)$.

Proposition 4.3.7. *The mapping $F \rightsquigarrow F \restriction (\mathrm{ON}{\leq}3k)$ defines a bijection from the set of strongly finite dilators of degree $\leq k$ onto the set of k-things.*

Proof. If F is a strongly finite dilator of degree $\leqslant k$, then its restriction to $\mathrm{ON}\leqslant 3k$ is obviously a k-thing. Conversely, assume that F is a k-thing. We show that there exists one and only one extension (still denoted by F) of F into a strongly finite dilator of degree $\leqslant k$. First observe that the notation $(t; i_0, \ldots, i_{n-1}; i)$ is still possible for the k-thing F, when $i \leqslant 3k$. The integer n is then always $\leqslant k$.

(i) In a first step we extend F into a functor from $\mathrm{ON}<\omega$ to itself. Define $F(p)$ to be the set of all notations $(z; i_0, \ldots, i_{n-1}; p)$, with $i_0 < \cdots < i_{n-1} < p$. $F(p)$ is ordered as follows: suppose that $a = (t; i_0, \ldots, i_{m-1}; p)$, $a' = (t'; i'_0, \ldots, i'_{m'-1}; p)$, then form $b = (t; j_0, \ldots, j_{m-1}; 3k)$, $b' = (t'; j'_0, \ldots, j'_{m'-1}; 3k)$, with $i_r < i'_s$ iff $j_r < j'_s$ for all r, s (this is possible since $m + m' \leqslant 3k$), then by definition $a \leqslant a'$ iff in $F(3k)$ $b \leqslant b'$. The relation $\leqslant$ thus defined is a linear order. We have to check reflexivity, antisymmetry, transitivity, linearity, for instance transitivity, if $a'' = (t''; i''_0, \ldots, i''_{m''-1}; p)$, then form $b'' = (t'', j''_0, \ldots, j''_{m''-1}; 3k)$, and also b, b', such that

$$i_r \leqslant i'_s \leftrightarrow j_r \leqslant j'_s, \qquad i_r \leqslant i''_s \leftrightarrow j_r \leqslant j''_s, \qquad i'_r \leqslant i''_s \leftrightarrow j'_r \leqslant j''_s \quad \text{for all } r, s$$

(this is possible because $m + m' + m'' \leqslant 3k$). Now if $a \leqslant a'$ and $a' \leqslant a''$, then $b \leqslant b'$, $b' \leqslant b''$, and by transitivity of the relation $\leqslant$ on $F(3k)$, $b \leqslant b''$, so $a \leqslant a''$. If one defines $F(f)$, when $f \in I(p, q)$ by $F(f)(z; i_0, \ldots, i_{n-1}; p) = (z; f(i_0), \ldots, f(i_{n-1}); q)$, then one has obviously defined a functor from $\mathrm{ON}<\omega$ into itself. $F(n)$ is a polynomial of degree k, and F commutes to pull-backs. One checks easily (in analogy with the checking of transitivity) that $f \leqslant g \to F(f) \leqslant F(g)$. Furthermore, this extension (with $F(n)$ of degree k) is obviously unique.

(ii) It sufficies to show that the extension of the functor F from $\mathrm{ON}<\omega$ to itself to ON is a dilator; this is a consequence of the next Proposition.

Proposition 4.3.8. *Let F be a functor from* $\mathrm{ON}<\omega$ *to* $\mathrm{ON}<\omega$ *such that:*

(i) *$F(n)$ is a polynomial of n;*

(ii) *F commutes to pull-backs;*

(iii) $f \leqslant g \to F(f) \leqslant F(g)$;

then F is the restriction to $\mathrm{ON}<\omega$ *of a strongly finite dilator.*

Proof. Define $F(x)$ by direct limits. All we need is to show that $F(x)$ is well founded. If $(t_n; i_n^0, \ldots, i_n^{p_n-1}; x)$ is a strictly decreasing sequence in $F(x)$, then since t_n ranges over a finite set (this comes easily from the fact that $F(n)$ is a polynomial of n, as in Proposition 4.3.3) then one can extract a subsequence such that t_n is constant. So one can assume that the given strictly decreasing sequence can be written $(t; i_n^0, \ldots, i_n^{p-1}; x)$. Let $N^{(2)}$ be the set of all pairs (n, m) such that $n < m$. We shall define a partition of $N^{(2)}$ in p subsets: if $n < m$, then, since $(t; i_n^0, \ldots, i_n^{p-1}; x) > (t; i_m^0, \ldots, i_m^{p-1}; x)$, define $f, g \in I(p, x)$ by $f(k) = i_n^k$, $g(k) = i_m^k$; then by condition (iii), $f \leqslant g \to F(f)(a) \leqslant G(g)(a)$. If one takes $a = (t; 0, \ldots, p-1; p)$, we conclude that $f \nleqslant g$, so one can define the partition P by $P(n, m) =$ smallest r such that $i_n^r > i_m^r$. Applying Ramsey's theorem, one gets an infinite subset X such that, for all $n, m \in X$, $n < m$, then $P(n, m) = r =$ constant.

The ordinals i_n^r, for n in X form a strictly decreasing sequence in x, contradiction. (Alternative method: one can use Theorem 3.2.5.)

Corollary 4.3.9. *There is a primitive recursive function which enumerates (the codes of) all strongly finite dilators.*

Proof. It is possible to enumerate all pairs (k, t), where t is a k-thing. Furthermore, it is immediate that the function which associates to (k, t) the code of the unique strongly finite dilator of degree $\leq k$ extending t is primitive recursive.

Theorem 4.3.10. *If F is a dilator, then there is a direct system (F_i, T_{ij}) with all F_i strongly finite, and such that*

$$F = \varinjlim{}^*(F_i, T_{ij}).$$

Proof. Let $I = \{i; i \subset \mathrm{rg}(F), i \text{ finite}\}$, and order I by inclusion. Observe that I is directed. Define F_i and $T_i \in I(F_i, F)$ by $\mathrm{rg}(T_i) = i$, then F_i is obviously strongly finite. Define $T_{ij} \in I(F_i, F_j)$ by $T_j T_{ij} = T_i$, then $(F, T_i) = \varinjlim(F_i, T_{ij})$ by Proposition 4.2.6.

Definition 4.3.11. The direct system constructed in the proof of Theorem 4.3.10 is the *canonical system of F*.

Definition 4.3.12. The following data define a category SFD:
objects: strongly finite dilators;
morphisms from F to G: the set $I(F, G)$.

4.4. Predilators

Definition 4.4.1. A *predilator* is a functor from ON into OL enjoying:
(i) F commutes to direct limits;
(ii) F commutes to pull-backs;
(iii) $f \leq g \to F(f) \leq F(g)$.

Definition 4.4.2. The following data define a category PIL:
objects: predilators;
morphisms from F to G: the set $I(F, G)$ of natural transformations from F to G.

Remark 4.4.3. Obviously (Proposition 2.3.10) DIL is a subcategory of PIL. All definitions and results of this section can be adapted, mutatis mutandis, to the category PIL; this is left to the reader. For instance, the crucial result of Theorem 4.3.10 is still true in PIL: any predilator is a direct limit of strongly finite dilators (if F_i is defined by $\mathrm{rg}(F_i) = i$, with i finite, $i \subset \mathrm{rg}(F)$, then F_i is a strongly finite dilator by Proposition 4.3.8). In fact Proposition 4.3.8 shows that strongly finite predilators are dilators.

The role played by PIL w.r.t. DIL is similar to the role played by OL w.r.t. ON. This analogy is enhanced by the following theorems.

Theorem 4.4.4. *In* PIL, *all direct systems have direct limits.*

Proof. Straightforward adaptation of Theorem 4.1.3.

Theorem 4.4.5. *Let ϕ be a functor from* SFD *into* OL *or* PIL, *then*

(i) *ϕ can be extended into a functor ψ from* PIL *into* OL *or* PIL, *commuting to direct limits.*

(ii) *If ϕ commutes to pull-backs, so does ψ.*

Proof. (i) is proved as Theorem 2.1.5(ii) is proved by copying Theorems 1.5.6 and 2.2.5.

Remark 4.4.6. (i) If ϕ is a functor from SFD to ON, and if the extension ψ has the property that $\psi(F)$ is well ordered for all dilators F, then ϕ can be extended into a functor from DIL into ON commuting to direct limits. This extension is of course unique.

(ii) If ϕ is a functor from SFD to DIL, and if the extension ψ has the property that $\psi(F)(x)$ is a well order for all dilators F and ordinals x, then ϕ can be extended into a functor from DIL into DIL commuting to direct limits. This extension is of course unique.

(iii) It is possible to apply Theorem 4.4.5 in the following situation: let H be a given dilator, and define a subcategory DIL$\leqslant H$ of DIL by taking as objects those F such that $I(F, H') \neq \emptyset$, for some H' which is either equal to H or a predecessor of H (Remark 3.5.2(i)). Let SFD$\leqslant H$ be the category SFD$\,\cap\,$DIL$\leqslant H$, then, since every dilator in DIL$\leqslant H$ can be expressed as a direct limit of dilators of SFD$\leqslant H$, it is possible to extend a given functor ϕ defined on SFD$\leqslant H$ to DIL$\leqslant H$, with the same properties as above.

5. The functor $\bigwedge$

5.1. *Some functors defined on* DIL

Theorem 5.1.1. *Define the functor* LH *from* DIL *to* ON *by:*

– *if $F = \sum_{x<a} F_x$, with all F_x perfect, then* LH$(F) = a$;

– *if $T = \sum_{x<f} T_x$, with $T_x \in I(F_x, G_{f(x)})$, F, G perfect, then* LH$(T) = f$.

(i) *the functor 'length' commutes to direct limits.*

(ii) LH *does not commute to pull-backs.*

Proof. (i) LH commutes to $\varinjlim$: this amounts to show that, given $z \in \mathrm{LH}(F)$, there exists T, with $\mathrm{rg}(T)$ finite and $z \in \mathrm{rg}(\mathrm{LH}(T))$. But, if $z \in \mathrm{LH}(F)$, choose a point $(n, (a; 0, \ldots, n-1; n))$ in the zth equivalence class modulo (see Theorem 3.1.5), then, if one defines T by $\mathrm{rg}(T) = \{(a; n)\}$ it is clear that $T \in I(G, F)$, for a prime G. So $\mathrm{LH}(G) = 1$, and $\mathrm{LH}(T)(0) = z$.

(ii) LH does not commute to pull-backs: for instance take $G = \mathrm{Id}^2$, and let us recall that $\mathrm{rg}(G) = \{(1, 2), (0, 1), (2, 2)\}$ (Example 3.3.9(ii)), and G is perfect, so $\mathrm{LH}(G) = 1$. Define T_1 by $\mathrm{rg}(T_1) = \{(1, 2)\}$ and T_2 by $\mathrm{rg}(T_2) = \{(0, 1)\}$, then $\mathrm{LH}(T_1) = \mathrm{LH}(T_2) = E_1$. But $\mathrm{rg}(T_1 \,\&\, T_2) = \mathrm{rg}(T_1) \cap \mathrm{rg}(T_2)$ by Theorem 4.2.7, so $\mathrm{rg}(T_1 \,\&\, T_2) = \emptyset$. This forces $\mathrm{LH}(T_1 \,\&\, T_2) = E_{01}$, distinct from $\mathrm{LH}(T_1) \,\&\, LH(T_2) = E_1$.

Theorem 5.1.2. *The functor 'composition' from* DIL×DIL *into* DIL, *defined as usual, commutes to direct limits and to pull-backs. Furthermore, if $T \leqslant T'$, $U \leqslant U'$, then $T \circ U \leqslant T' \circ U'$.*

Proof. $\circ$ commutes to direct limits: it suffices to show that, given any point $(a; n)$ in $\mathrm{rg}(F \circ G)$, there exist $T \in I(F', F)$, $U \in I(G', G)$, with $\mathrm{rg}(T)$, $\mathrm{rg}(U)$ finite, and $(a; n) \in \mathrm{rg}(T \circ U)$. Let H be $F \circ G$, and let us use the subscripts $(\cdots)_F, (\cdots)_G, (\cdots)_H$ to distinguish between notations relative to F, G or H; as usual:

$$a = (a; 0, \ldots, n-1; n)_H = (b; x_0, \ldots, x_{m-1}; G(n))_F,$$

$$x_i = (c_i; p_i^0, \ldots, p_i^{k_i-1}; n)_G.$$

If one defines T by $\mathrm{rg}(T) = \{(b; m)\}$ and U by $\mathrm{rg}(U) = \{(c_i; k_i); i = 0, \ldots, m-1\}$, then $(a; n) \in \mathrm{rg}(T \circ U)$.

$\circ$ commutes to pull-backs: if $H = F \circ G$, then $(a; n) \in \mathrm{rg}(T \circ U)$ iff $(b; m) \in \mathrm{rg}(T)$ and $(c_i; k_i) \in \mathrm{rg}(U)$ for all $i \leqslant m-1$ (with the definitions given above for $(b; m)$ and $(c_i; k_i)$). From this, commutation to pull-backs is immediate.

$\circ$ preserves $\leqslant$: immediate, left to the reader.

Theorem 5.1.3. *Define the functor 'iteration'* IT *by:*

– $\mathrm{IT}(F)(x) = G(x, x)$, *where G is defined by Example* 2.4.9(iii);

$$\mathrm{IT}(F)(f) = G(f, f)$$

– *if $T \in I(F, F')$, then define G, G' as in Example* 2.4.9(iii), *and a natural transformation U from G to G' by:*

$$U(x, 0) = E_x, \qquad U(x, y+1) = U(x, y) + T(U(x, y)),$$

$$U(x, \sup(y_i)) = \bigcup_i U(x, y_i)$$

and let $\mathrm{IT}(T)(x) = U(x, x)$.

IT *commutes to direct limits and to pull-backs.*

Proof. IT commutes to direct limits: if $(a;n)\in \mathrm{rg}(H)$, with $H=\mathrm{IT}(T)$, then, define a function f from $G(n,n)$ into the set of finite subsets of $\mathrm{rg}(F)$:
- if $x<G(n,0)=n$, then $f(x)=\emptyset$;
- if $x=G(n,p)+(b;x_0,\ldots,x_{k-1};G(n,p))_F$, then

$$f(x)=\{(b;k)\}\cup f(x_0)\cup\cdots\cup f(x_{k-1}).$$

If $(a;n)\in \mathrm{rg}(H)$, and if $T\in I(F',F)$ is defined by $\mathrm{rg}(T)=f(a)$, then it is immediate that $(a;n)\in \mathrm{rg}(\mathrm{IT}(T))$.

IT commutes to pull-backs: if f is defined as above, then it is clear that $(a;n)\in \mathrm{rg}(\mathrm{IT}(T))$ iff $f(a)\subset \mathrm{rg}(T)$. From this, if $T_3=T_1\mathbin{\&}T_2$, then $(a;n)\in \mathrm{rg}(\mathrm{IT}(T_3))$ iff $f(a)\subset \mathrm{rg}(T_3)=\mathrm{rg}(T_1)\cap \mathrm{rg}(T_2)$, i.e., iff $f(a)\subset \mathrm{rg}(T_1)$ and $f(a)\subset \mathrm{rg}(T_2)$, i.e., iff $(a;n)\in \mathrm{rg}(\mathrm{IT}(T_1))\cap \mathrm{rg}(\mathrm{IT}(T_2))$.

Remark 5.1.4. Many other functors already constructed are functors from DIL to DIL commuting to direct limits and pull-backs, and hence preserving $\leqslant$: for instance Example 2.4.9(i) defines a functor $\int$ from DIL into DIL with these properties: if F is a dilator, then $\int(F)$ is the dilator G defined by Example 2.4.9(i); if $T\in I(F,F')$, then $\int(T)$ is defined by:

$$\int(T)(x)=\sum_{y<x} T(y).$$

5.2. Functors involving ΩDIL

Remark 5.2.1. The category ΩDIL is obviously a subcategory of DIL. It would be fastidious to rewrite those properties of DIL which still hold for ΩDIL: they are very easy to check. The isomorphisms UN and SEP allow us to identify the category ΩDIL with the category BIL. We shall do this systematically. The interest of the isomorphism is that it enables us to make sharper constructions, for instance, semi-composition, or semi-iteration below. The price to pay is that the isomorphisms SEP and UN are not so easy to handle.

Definition 5.2.2. If F is a bilator, one defines $\mathrm{Rg}(F)$ to the set of all triplets $(a;n;m)$ such that $a=(a;0,\ldots,n-1;n;0,\ldots,m-1)$ in the notation system deduced from F by Remark 3.4.4(iii). If F, G are bilators, if T is a natural transformation from F to G, one defines $\mathrm{Rg}(T)$ by:

$$(a;n;m)\in \mathrm{Rg}(T)\quad \text{iff } (a;n;m)\in \mathrm{Rg}(G)\quad \text{and}\quad a\in \mathrm{rg}(T(n,m)).$$

Remarks 5.2.3. (i) Since we have decided to identify dilators of kind Ω and bilators, we have two definitions of range for these objects: rg and Rg.

(ii) The reader will admit without justification the analogues of most of the properties of section 4, for bilators, and in terms of Rg.

Theorem 5.2.4. *The functor 'semi-composition' from ΩDIL$\times\Omega$DIL to ΩDIL:*

$$(F\circ_s G)(x,y)=F(x,G(x,y)),\qquad (F\circ_s G)(f,g)=F(f,G(f,g)),$$
$$(T\circ_s U)(x,y)=T(x,U(x,y)),$$

commutes to pull-backs and direct limits, and preserves $\leqslant$.

Proof. $\circ_s$ preserves $\leqslant$: immediate.

$\circ_s$ commutes to direct limits: $\circ_s$ in fact defined as a functor from BIL$\times$BIL to BIL. By general properties of isomorphisms, it is sufficient to show that $\circ_s$ (viewed as a functor from BIL^2 to BIL) commutes to direct limits (similarily for pull-backs, see below). This amounts to show that given $(a;n;m)\in \mathrm{Rg}(F\circ_s G)$, there exist strongly finite bilators F' and G', $T\in I(F',F)$, $U\in I(G',G)$, such that $(a;n;m)\in\mathrm{Rg}(T\circ_s U)$, with $H=F\circ_s G$. Write:

$$a=(a;0,\dots,n-1;n;0,\dots,m-1)_H$$
$$\quad=(b;u_0,\dots,u_{k-1};n;v_0,\dots,v_{l-1})_F,$$

and
$$v_i=(a^i;u_0^i,\dots,u^i_{p_i-1};n;v_0^i,\dots,v^i_{q_i-1})_G,\quad \text{if } \mathrm{Rg}(T)=\{(b;k;l)\}$$
$$\mathrm{Rg}(U)=\{(a^i;p_i;q_i);\ i=0,\dots,l-1\},$$

then obviously $(a;n;m)\in\mathrm{Rg}(T\circ_s U)$.

$\circ_s$ commutes to pull-backs: with the notations above, it is clear that $(a;n;m)\in \mathrm{Rg}(T_i\circ_s U_i)$ iff $(a;k;l)\in\mathrm{Rg}(T_i)$ and $(a^i,p_i,q_i)\in\mathrm{Rg}(U_i)$ for $i<l$ from this it is immediate that, if $\mathrm{Rg}(T_1)\cap\mathrm{Rg}(T_2)=\mathrm{Rg}(T_3)$, if $\mathrm{Rg}(U_1)\cap\mathrm{Rg}(U_2)=\mathrm{Rg}(U_3)$, then

$$\mathrm{Rg}(T_1\circ_s U_1)\cap\mathrm{Rg}(T_2\circ_s U_2)=\mathrm{Rg}(T_3\circ_s U_3).$$

Remark 5.2.5. If F is a bilator, then $\mathrm{Rg}(F)$ and $\mathrm{rg}(\mathrm{UN}(F))$ are equipotent: this is easily seen from the definition. Hence the word 'strongly finite' has the same meaning when F is considered as a bilator, or when F is considered as a dilator of kind Ω.

5.3. Generalized semi-products

Notations 5.3.1. We define $E_{FG}\in I(F,G)$, when $G=F+F'$ for some F': $E_{FG}=F+T'$, where F $(=E_{FF})$ is the identity of F, and T' $(=E_{\underset{\sim}{0}F'})$ is defined by $T'\in I(\underset{\sim}{0},F')$. We shall abbreviate E_{FF} in E_F.

Lemma 5.3.2. *Assume that (F_i,T_{ij}) is a direct system in ΩDIL, such that: for all i, j, $i<j\to F_j=F_i\circ_s F_{ij}$ for some $F_{ij}\in\Omega$DIL, and $T_{ij}=E_{F_i}\circ_s U_{ij}$, where $U_{ij}\in I(\mathrm{Id},F_{ij})$ enjoys: $U_{ij}(x,y)(z)=F_{ij}(x,z)$ for all x, y and $z<y$, then*

(i) *(F_i,T_{ij}) has a direct limit (F,T_i) in ΩDIL.*

(ii) *If $z<y$, then $T_i(x,y)(F_i(x,z))=F(x,z)$*

(iii) *If $C_i(x)=\{F_i(x,y);\ y\in\mathrm{ON}\}$, if $C(x)=\{F(x,y);\ y\in\mathrm{ON}\}$, then $C(x)=\bigcap_{i\in I}C_i(x)$.*

Proof. (i) Let us work in the category BIL. By the obvious analogue of Theorem 4.1.3(i) for BIL, it is sufficient to show that the direct system $(F_{ij}(x, y), T_{ij}(x, y))$ admits a direct limit in ON for all $x, y \in \mathrm{ON}$. We define by induction strictly increasing functions f_{ix} from ON to ON, by:

$$f_{ix}(0) = \sup_{j>i} F_{ij}(x, 0); \qquad f_{ix}(y+1) = \sup_{j>i} F_{ij}(x, f_{jx}(y)+1);$$

$$f_{ix}(y) = \sup_{t<y} f_{ix}(t), \quad \text{if } y \text{ is limit.}$$

We show by induction on y that

$$i<j \to f_{ix}(y) = F_{ij}(x, f_{jx}(y)): \quad f_{ix}(0) = \sup_{k>j} (F_{ij}(x, F_{jk}(x, 0))),$$

but $F_{ij}(x, z)$ is continuous in z (Theorem 2.4.3), hence

$$f_{ix}(0) = F_{ij}\Big(x, \sup_{k>j} (F_{jk}(x, 0))\Big) = F_{ij}(x, f_{jx}(0)).$$

The case y successor is handled in a similar way, the case y limit being trivial.

We shall define functions g_{ixy} from $F_i(x, y)$ to ON by: if

$$z_i = (a; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1})_{F_i},$$

then

$$g_{ixy}(z_i) = (a; x_0, \ldots, x_{n-1}; x; f_{ix}(y_0), \ldots, f_{ix}(y_{n-1}))_{F_i},$$

i.e., $g_{ixy}(z_i) = F_i(E_x, f_{ix})(z_i)$, so g_{ixy} is strictly increasing. Moreover, if $z_j = T_{ij}(x, y)(z_i)$, then $z_j = (b; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1})_{F_j}$, and also

$$z_j = (a; x_0, \ldots, x_{m-1}; x; F_{ij}(x, y_0), \ldots, F_{ij}(x, y_{n-1}))_{F_i}$$

(this last equality comes from the hypothesis of the lemma), then

$$\begin{aligned} g_{jxy}(z_j) &= (b; x_0, \ldots, x_{m-1}; x; f_{jx}(y_0), \ldots, f_{jx}(y_{n-1}))_{F_j} \\ &= (a; x_0, \ldots, x_{m-1}; x; F_{ij}(x, f_{jx}(y_0)), \ldots, F_{ij}(x, f_{jx}(y_{n-1})))_{F_i} \\ &= (a; x_0, \ldots, x_{m-1}; x; f_{ix}(y_0), \ldots, f_{ix}(y_{n-1}))_{F_i} = g_{ixy}(z_i), \end{aligned}$$

so we have shown that $g_{ixy} = g_{jxy} T_{ij}(x, y)$. Then, applying Corollary 1.4.4, it follows that the system $(F_i(x, y), T_{ij}(x, y))$ has a direct limit in ON.

(ii) Observe that, as a consequence of the hypothesis of Lemma 5.3.2

$$i<j \to T_{ij}(x, y)(F_i(x, z)) = F_j(x, z) \quad (\text{if } z<y).$$

We have $(F(x, y), T_i(x, y)) = \varinjlim(F_i(x, y), T_{ij}(x, y))$ for all x, y. Choose i such that $F(x, z) \in \mathrm{rg}(T_i(x, z+\omega))$, so $F(x, z) = T_i(x, z+\omega)(a_i(x, z))$; $F(x, z)$ is the smallest object $(a; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1})_F$ with $n \neq 0$ and $y_{n-1} \geqslant z$, then $a_i(x, z)$ is the smallest object $(b; x_0, \ldots, x_{m-1}; x; y_0, \ldots, y_{n-1})_{F_i}$, with $n \neq 0$ and $y_{n-1} \geqslant z$, i.e., $a_i(x, z) = F_i(x, z)$ (see Remark 2.4.10(iv). We have shown that $T_i(x, y)(F_i(x, y)) = F(x, y)$, and by the remark above, this will be true for i arbitrary.

(iii) Let $C(x, z)$ be the zth element of $\bigcap_i C_i(x)$. We show that $F(x, z) = C(x, z)$

by proving the double inequality:

$$(1)\qquad F(x,z)=\varinjlim_{I}{}^{*}(F_i(x,z),T_{ij}(x,z))$$
$$=\varinjlim_{K}{}^{*}(F_k(x,F_{ki}(x,z)),F_k(E_x,F_{ki}(E_x,U_{ij}(x,z))))$$
$$=F_k\Big(x,\varinjlim_{K}{}^{*}(F_{ki}(x,z),F_{ki}(E_x,U_{ij}(x,z)))\Big),$$

with $K=\{i; i\in I \text{ and } i\succ k\}$ so $F(x,z)\in C_i(x)$ for all i, hence $F(x,z)\in C(x)$, so $F(x,z)\geqslant C(x,z)$.

(2) Conversely, we show by induction on z that $C(x,z)=F_i(x,f_{ix}(z))$, for all $i\in I$. First observe that,

$$i\prec j\rightarrow F_i(x,f_{ix}(z))=F_i(x,F_{ij}(x,f_{jx}(z)))=F_j(x,f_{jx}(z)).$$

From this it follows that the value $F_i(x,f_{ix}(z))$ is independent of i, and it will be sufficient to prove the property for one $i\in I$. The case $z=0$ and z limit are left to the reader. We prove the property for $z+1$: if, for all $i\in I$ $C(x,z)=F_i(x,f_{ix}(z))$, then observe that $i\prec j\rightarrow f_{ix}(z)\geqslant f_{jx}(z)$. (Indeed, $f_{ix}(z)=F_{ij}(x,f_{jx}(z))$ and $F_{ij}(x,\cdot)$ is strictly increasing.) Hence there is an index k such that, for $i\succ k$, $f_{ix}(z)$ has a constant value t, and we have, if $k\prec i\prec j$:

$$t=f_{ix}(z)=F_{ij}(x,f_{jx}(z))=F_{ij}(x,t).$$

Let $D_k(x)=\bigcap_{j\succ k}\mathrm{rg}(F_{kj}(x,\cdot))$, then obviously

$$C(x)=\{v; v=F_k(x,u)\text{ for some } u\in D_k(x)\}.$$

From this it follows that $C(x,z+1)=F_k(x,u)$, where u is the smallest element in $D_k(x)$ strictly greater than t. The expression of u is familiar:

$$u=\sup_{j\succ k}F_{kj}(x,t+1)=\sup_{j\succ k}F_{kj}(x,f_{jx}(z)+1)=f_{kx}(z+1).$$

Hence $C(x,z+1)=F_k(x,f_{kx}(z+1))$.

Now observe that $T_i(x,y)(F_i(x,z))\leqslant g_{ixy}(F_i(x,z))$. Since by definition of g_{ixy} one has $g_{ixy}(F_i(x,z))=F_i(x,f_{ix}(z))$, one gets

$$F(x,z)=T_i(x,y)(F_i(x,z))\leqslant g_{ixy}(F_i(x,z))=F_i(x,f_{ix}(z))=C(x,z).$$

Remarks 5.3.3. (i) The reader has recognized in Lemma 5.3.2 the functorial analogue of the well-known property of *normal* (i.e., strictly increasing and continuous) functions from ON to ON: if $(g_i)_{i\in I}$ is a family of normal functions such that $i\prec j\rightarrow \mathrm{rg}(g_j)\subset\mathrm{rg}(g_i)$, then it is possible to construct a new normal function by

$$\mathrm{rg}(g)=\bigcap_{i\in I}\mathrm{rg}(g_i).$$

(ii) The reader has surely noticed that in Lemma 5.3.2 the variable x plays no role; in fact we are working on flowers, and the natural formulation of Lemma

5.3.2 is in the category of flowers, forgetting the extra variable x. The same remark holds for the next definitions, for instance, regularity could be directly defined for flowers, and one should define the *generalized product* of a family of flowers... By the way, notice the exact correspondance between regularity (for a flower) and Remark 2.4.12.

(iii) Use of these technics can be found in the papers [9, 10].

Definition 5.3.4. (i) The bilator F is *regular* iff

$$\forall x, x', y, y', z \in \mathrm{ON}\, \forall f \in I(x, x')\, \forall g \in I(y, y')$$
$$(z < y \to F(x, z) < F(x, y) \quad \text{and} \quad F(f, g)(F(x, z)) = F(x', g(z))).$$

(ii) If F, G are regular bilators, if $T \in I(F, G)$, then T is *regular* iff

$$\forall x, y, z \in \mathrm{ON}(z < y \to T(x, y)(F(x, z)) = G(x, z)).$$

For instance, if F is a regular bilator, then $\mu_F \in I(\mathrm{Id}, F)$ defined by $\mu_F(x, y)(z) = F(x, z)$ is regular.

(iii) BIL_r is a category, defined by:

objects: regular bilators;

morphisms from F to G: the set $I_r(F, G)$ of regular morphisms from F to G.

Definition 5.3.5. If $(F_t)_{t<x}$ is a family of regular bilators, then one defines a new regular bilator $\prod_{t<x} F_t$, the *semi-product* of the family (F_t). If $(G_u)_{u<y}$ is another such family, if $f \in I(x, y)$, if $T_t \in I_r(F_t, G_{f(t)})$ for all $t < x$, then one defines

$$\prod_{t<f} T_t \in I_r\Big(\prod_{t<x} F_t, \prod_{u<y} G_u\Big),$$

as follows (we proceed by induction on y):

(i) if $x = y = 0$, then

$$\prod_{t<x} F_t = \prod_{u<y} G_u = \mathrm{Id}, \qquad \prod_{t<f} T_t = E_{\mathrm{Id}},$$

(ii)

$$\prod_{u<y+1} G_u = \Big(\prod_{u<y} G_u\Big) \circ_s G_y, \qquad \prod_{t<f+E_1} T_t = \Big(\prod_{t<f} T_t\Big) \circ_s T_x,$$
$$\prod_{t<f+E_{01}} T_t = \Big(\prod_{t<f} T_t\Big) \circ_s \mu_{G_y},$$

(iii)

$$\prod_{u<y} G_u = \varinjlim_{y'<y}{}^{*}\Big(\prod_{u<y'} G_u, \prod_{u<E_{y'y}} E_{G_u}\Big), \qquad \prod_{t<\bigcup_i f_i} T_t = \varinjlim_i\Big(\prod_{t<f_i} T_t\Big)$$

when y is limit.

Example 5.3.6. Take $x=3$, $y=7$, $f(0)=2$, $f(1)=3$, $f(2)=5$, then

$$\begin{array}{rcccccccccccccc}
t<\prod_3 F_t = & \mathrm{Id} & \circ_s & \mathrm{Id} & \circ_s & F_0 & \circ_s & F_1 & \circ_s & \mathrm{Id} & \circ_s & F_2 & \circ_s & \mathrm{Id} \\
& | && | && | && | && | && | && | \\
t<\prod_f T_t = & \mu_{G_0} & \circ_s & \mu_{G_1} & \circ_s & T_0 & \circ_s & T_1 & \circ_s & \mu_{G_4} & \circ_s & T_2 & \circ_s & \mu_{G_6} \\
& \downarrow && \downarrow && \downarrow && \downarrow && \downarrow && \downarrow && \downarrow \\
t<\prod_6 G_t = & G_0 & \circ_s & G_1 & \circ_s & G_2 & \circ_s & G_3 & \circ_s & G_4 & \circ_s & G_5 & \circ_s & G_6
\end{array}$$

Proposition 5.3.7. *Definition* 5.3.5 *is sound.*

Proof. We show, by induction on y, that $\prod_{u<y} G_u$ and $\prod_{t<f} T_t$ (if $f \in I(x, y)$) exist. We show also:

(P): if $y = y' + y''$, $f = f' + f''$,

$$\prod_{u<y} G_u = \prod_{u<y} G_u \circ_s \prod_{u<y''} G_{y'+u} \quad \text{and} \quad \prod_{t<f} T_t = \prod_{t<f'} T_t \circ_s \prod_{t<f''} T_{x'+t}.$$

(i) if $y = 0$, everything is trivial;

(ii) if the properties hold for y, then they hold for $y+1$;

(iii) if y is limit, then the system $(\prod_{u<y'} G_u, \prod_{u<E_{y'y''}} E_{G_u})y' < y'' < y$ enjoys the hypotheses of Lemma 5.3.2: This is an immediate consequence of (P). From this the existence of the limit $\varinjlim{}^*(\prod_{u<y} G_u, \prod_{u<E_{y'y}} E_{g_u})$ is ensured.

From this, the direct limit $\varinjlim(\prod_{t<f_i} T_t)$ exists.

Now, we prove (P): if $y = y' + y''$, then, if one forgets the trivial case $y'' = 0$, one can write

$$\begin{aligned}
\prod_{u<y} G_u &= \varinjlim_{z<y'}{}^*\Big(\prod_{u<y'+z} G_u, \prod_{u<E_{y'}+E_{zz'}} E_{G_u}\Big) \\
&= \varinjlim_{z<y'}{}^*\Big(\prod_{u<y'} G_u \circ_s \prod_{u<z} G_{y'+u}, \prod_{u<E_{y'}} E_{G_u} \circ_s \prod_{u<E_{zz'}} E_{G_{y'+u}}\Big) \\
&= \prod_{u<y'} G_u \circ_s \varinjlim_{z<y''}{}^*\Big(\prod_{u<z} G_{y'+u}, \prod_{u<E_{zz'}} E_{G_{y'+u}}\Big) = \prod_{u<y'} G_u \circ_s \prod_{u<y''} G_{y'+u}.
\end{aligned}$$

The formula for the natural transformations is proved in the same way.

Proposition 5.3.8. (i) *Assume that* $(x, f_i) = \varinjlim_I (x_i, f_{ij})$, *and for* $l < x$, *let* $I_l = \{i; l \in \mathrm{rg}(f_i)\}$, *and let* (F_i^l, T_{ij}^l) *and* (F^l, T_i^l) *in* BIL, *by such that* $\varinjlim_{I_l}(F_i^l, T_{ij}^l) = (F^l, T_i^l)$. *Define*

$$(G_i^l)_{l<x_i}, U_{ij}^l \in I_r(G_i^l, G_j^f ij^{(l)}) \quad \textit{and} \quad U_i^l \in I_r(G_i^l, F^{f_i(l)}),$$

by: $G_i^l = F_i^{f_i(l)}$, $U_{ij}^l = T_{ij}^{f_i(l)}$, $U_i^l = T_i^{f_i(l)}$. *Then, one shows easily that*

$$\Big(\prod_{l<x} F^l, \prod_{l<f_i} U_i^l\Big) = \varinjlim_I\Big(\prod_{l<x_i} G_i^l, \prod_{l<f_{ij}} U_{ij}^l\Big).$$

Assume now that x', I', x'_i, f'_i, f'_{ij}, I'_i, F'^l, T'^l_i, F'^l_i, T'^l_{ij} *are as above, let* (h_i) *be a direct system from* (x_i, f_{ij}) *to* (x'_i, f'_{ij}), *with* $h = \varinjlim(h_i)$. *Assume that, for all* $l < x$, $(V^l_i)_{i \in I_l}$ *is a direct system of morphisms from* $(F^l_i, T^l_{ij})_{I_l}$ *to* $(F'^l_i, T'^l_{ij})_{I'_l}$ *and that* $V^l = \varinjlim_{I_l}(V^l_i)$. *Let* $W^l_i = V^{f_i(l)}_i$ *for* $l < x_i$, *then we have:*

$$\prod_{l<h} V^l = \varinjlim_{I}\Big(\prod_{l<h} W^l_i\Big).$$

(ii) *Assume that* $T^l_i \in I_r(F^l_i, G^{f_i(l)}_i)$ *for* $i = 1, 2$ *and* $l < x_i$, *and* $f_i \in I(x_i, y)$, *then*

$$\prod_{l<f_1} T^l_1 \mathbin{\&} \prod_{l<f_2} T^l_2 = \prod_{l<f_1 \& f_2} (T^l_1 \mathbin{\&} T^l_2).$$

Proof. These properties are left as exercises for the reader. Observe that, in BIL_r, contrarily to BIL, pull-backs always exist.

Remark 5.3.9. Proposition 5.3.8 expresses exactly that '$\prod$ commutes to direct limits and to pull-backs'. This could be made more precise as follows: it is possible to define a category whose objects are sequences of regular bilators $(F_i)_{i<x}$, and morphisms are families $(T_i)_{i<x}$, such that $T_i \in I_r(F_i, G_{f(i)})$ for some $f \in I(x, y)$; then it is easy to prove a characterization theorem of direct limits and pull-backs in this new category, in such a way that Proposition 5.3.8 can be exactly read as '$\prod$ commutes to direct limits and to pull-backs'. This is left to the reader. Observe that with exactly the same construction, replacing BIL_r by DIL, $\sum$ appears as a functor commuting to $\varinjlim$ and &.

5.4. *The functor* $\bigwedge$

Definition 5.4.1. One defines a functor $\bigwedge$ from DIL to BIL_r, as follows:

(i) If $G = \sum_{t<y} G_t$, with G_t perfect for all t, then

$$\bigwedge G = \prod_{t<y} \bigwedge G_t.$$

If $T \in I(F, G)$, m and $F = \sum_{t<x} F_t$, with F_t perfect for all t, if $f \in I(x, y)$ and the family $T_t \in I(F_t, G_{f(t)})$ is such that $T = \sum_{t<f} T_t$, then

$$\bigwedge T = \prod_{t<f} \bigwedge T_t.$$

(ii) $\bigwedge \underline{1} = \mathrm{Id} + \mathrm{Id}$ (so $\bigwedge E_{\underline{1}} = E_{\mathrm{Id}+\mathrm{Id}}$).

(iii) If G is perfect and $\neq \underline{1}$, if $f \in I(x, x')$, $g \in I(y, y')$, then write

$$\mathrm{SEP}(G)(\cdot, y) = \sum_{t<y} {}_tG, \qquad \mathrm{SEP}(G)(\cdot, g) = \sum_{t<g} {}_{gt}G,$$

with ${}_{gt}G \in I({}_tG, {}_{g(t)}G)$. With the notations of Section 3.6, ${}_{gt}G$ is exactly ${}_{g'}G$, with $g' \in I(t, g(t))$ defined by $g'(z) = g(z)$ for all $z < t$; then

$$(\bigwedge G)(x, y) = \Big(\prod_{t<y} (\underline{1} + \bigwedge_t G)\Big)(x, 0),$$

$$(\bigwedge G)(f, g) = \Big(\prod_{t<g} (E_1 + \bigwedge_{gt} G)\Big)(f, E_0).$$

If F is perfect too, if $T \in I(F, G)$, then assume that $\mathrm{SEP}(T)(\cdot, y) = \sum_{t<E_y} {}_tT$, with ${}_tT \in I({}_tF, {}_tG)$; then

$$(\bigwedge T)(x, y) = \Big(\prod_{t<E_y} (E_1 + \bigwedge_t T)\Big)(x, 0).$$

Remarks 5.4.2. (i) It is obvious that $\bigwedge$ is some kind of exponential, since $\bigwedge$ transforms sums into products. The relations of $\bigwedge$ with the usual exponential will be investigated in part II.

(ii) Another connotation is that $\bigwedge$ is something like the Bachmann hierarchy. The precise relation is of no interest for the paper here; the reader will find answers in [10] and [18].

Theorem 5.4.3. *Definitions* 5.4.1 *is sound; more precisely, the functor* $\bigwedge$ *is uniquely defined by the conditions* 5.4.1(i)–(iii). *Furthermore, it satisfies the following properties, equivalent to* 5.4.1(i)–(iii):

(i) $\bigwedge$ *is a functor from* DIL *to* BIL; *if* G *is a dilator,* $\bigwedge G$ *is a bilator, if* $T \in I(F, G)$, *then* $\bigwedge T \in I(\bigwedge F, \bigwedge G)$, $\bigwedge E_G = E_{\bigwedge G}$, *and* $\bigwedge(TU) = (\bigwedge T)(\bigwedge U)$.

(ii) $\bigwedge \underline{0} = \mathrm{Id}$, $\bigwedge E_0 = E_{\mathrm{Id}}$.

(iii) $\bigwedge \underline{1} = A$, $\bigwedge E_1 = E_A$, *with* $A = \mathrm{Id} + \mathrm{Id}$.

(iv) *If* $T \in I(F, G)$, *with* F, G *perfect, then*

$$(\bigwedge G)(x, y) = \Big(\prod_{t<y} (\underline{1} + \bigwedge_t G)\Big)(x, 0),$$

$$(\bigwedge G)(f, g) = \Big(\prod_{t<g} (E_1 + (\bigwedge_{gt} G)\Big)(f, E_0),$$

$$(\bigwedge T)(x, y) = \Big(\prod_{t<E_y} (\underline{1} + \bigwedge_{gt} T)\Big)(x, 0).$$

(v) $\bigwedge(F' + F'') = (\bigwedge F') \circ_s (\bigwedge F'')$.

$\bigwedge(T' + T'') = (\bigwedge T') \circ_s (\bigwedge T'')$.

(vi) $(\bigwedge E_{0F})(x, y)(z) = (\bigwedge F)(x, z)$.

(vii) $\bigwedge$ *commutes to direct limits.*

(viii) $\bigwedge$ *commutes to pull-backs.*

Proof. Suppose that we have obtained $\bigwedge$ enjoying (i)–(viii), then $\bigwedge$ is a solution of Definition 5.4.1.

5.4.1(i): By induction on y. If $y=0$, apply (ii). If $y=y'+1$, then $G=G'+G_{y'}$, with $G'=\sum_{t<y'} G_t$, then, by induction hypothesis, $\bigwedge G'=\prod_{t<y'} \bigwedge G_t$, so, by (v), one gets:

$$\bigwedge G=\prod_{t<y'} \bigwedge G_t \circ_x \bigwedge G_{y'}=\prod_{t<y} \bigwedge G_t.$$

The case of morphisms is similar. If y is limit, then using (vii), one gets

$$\bigwedge G=\varinjlim_{y'<y}{}^*\Big(\bigwedge \sum_{t<y'} G_t, \bigwedge E_{\sum_{t<y'}G_t\sum_{t<y''}G_t}\Big);$$

using the induction hypothesis, this limit is equal to

$$\varinjlim_{y'<y}{}^*\Big(\prod_{t<y'} \bigwedge G_t, \prod_{t<E_{y'y''}} \bigwedge E_{G_t}\Big)=\prod_{t<y} \bigwedge G_t.$$

The case of morphisms is similar.

5.4.1(ii) is exactly (iii).

5.4.1(iii) is exactly (iv).

Finally, observe that condition (vi) ensures that $\bigwedge$ is a functor from DIL into BIL_r (and not only into BIL): $(\bigwedge E_{0F})(x, y)(z)=(\bigwedge F)(x, z)$ implies that $\bigwedge E_{0F}=\mu_{\bigwedge F}$, i.e., that $\bigwedge F$ is regular. If $T\in I(F, G)$, then $TE_{0F}=E_{0G}$, and so this means that (by (i)) $\bigwedge T\mu_{\bigwedge F}=\mu_{\bigwedge G}$: so $\bigwedge T$ is regular.

The unicity of the functor defined by Definition 5.4.1 is immediate.

It remains to construct $\bigwedge$ satisfying (i)–(viii). We shall proceed as follows: given a dilator H, define a subcategory DIL$<H$, as follows: F is an object of DIL$<H$ iff $I(F, H')\neq\emptyset$, for some predecessor H' of H. Define DIL$\leqslant H=$ DIL$<(H+\underset{\sim}{1})$, then we show, by induction on H (Theorem 3.5.1) that there exists one and only one functor $\bigwedge$ from DIL$\leqslant H$ to ΩDIL enjoying (i)–(viii). The induction hypothesis is therefore the existence of an unique functor $\bigwedge$ from DIL$<H$ to ΩDIL enjoying (i)–(viii). If F is in DIL$\leqslant H$, let us denote by $h(F)$ the smallest H', for the predecessor relation, such that $I(F, H')\neq\emptyset$, H' varying through predecessors of H and H. The notation $h(F)<h(G)$ will mean that $h(F)$ is a predecessor of $h(G)$. We shall also allow the notation $h(F)\leqslant h(G)$, to mean $h(F)<h(G)$ or $h(F)=h(G)$. If $T\in I(F, G)$, let $h(T)=h(G)$.

The proof is divided in five cases.

Case 5.4.4. If H is of kind $\underset{\sim}{0}$, then $H=\underset{\sim}{0}$, and DIL$\leqslant\underset{\sim}{0}$ consists of one object: $\underset{\sim}{0}$, and one morphism: $E_{\underset{\sim}{0}}$; if one defines

$$\bigwedge\underset{\sim}{0}=\mathrm{Id},\qquad \bigwedge E_{\underset{\sim}{0}}=E_{\mathrm{Id}}.$$

then (i)–(viii) are obviously satisfied.

Case 5.4.5. If $H=\underset{\sim}{1}$, then there are only two objects ($\underset{\sim}{0}$ and $\underset{\sim}{1}$) and three morphisms ($E_{\underset{\sim}{0}}, E_{\underset{\sim}{0}\underset{\sim}{1}}, E_{\underset{\sim}{1}}$). Define $\bigwedge\underset{\sim}{1}$, $\bigwedge E_{\underset{\sim}{0}\underset{\sim}{1}}$, $\bigwedge E_{\underset{\sim}{1}}$ as in (iii) and (vi);

(i): $(\bigwedge T)(\bigwedge U)=\bigwedge(TU)$; assume $h(T)=1$, then either $T=E_{\underset{\sim}{1}}$, hence $TU=U$, so $(\bigwedge T)(\bigwedge U) = E_A(\bigwedge U) = (\bigwedge U) = \bigwedge(TU)$, or $U = E_0$, and then $TU = T$, so $(\bigwedge T)(\bigwedge U) = (\bigwedge T)(E_{\mathrm{Id}}) = (\bigwedge T) = \bigwedge(TU)$

(ii)–(iv): Trivial.

(v): If $F = F' + F''$ is in DIL$\leqslant \underline{1}$, then one of F' and F'' is $\underline{0}$, then one of $\bigwedge F'$ and $\bigwedge F''$ is Id, and $\bigwedge F = \bigwedge F' \circ_s \bigwedge F''$, because Id is neutral for the composition law $\circ_s$. If $T = T' + T''$, then $T' = E_0$, or $T'' = E_0$, so $\bigwedge T' = E_{\mathrm{Id}}$ or $\bigwedge T'' = E_{\mathrm{Id}}$, so $\bigwedge T = \bigwedge T' \circ_s \bigwedge T''$, as above.

(vi): By definition, when $h(F) = 1$, $(\bigwedge E_{01})(x, y)(z) = A(x, z)$. This definition is possible because of our choice of $A = \mathrm{Id} + \mathrm{Id}$.

(vii): Trivial.

(viii): If $T_1 \,\&\, T_2 = T_3$, with $h(T_i) = \underline{1}$, then:

– if $T_1 = E_1$, then $T_2 = T_3$, and $\bigwedge T_1 \,\&\, \bigwedge T_2 = E_A \,\&\, \bigwedge T_2 = \bigwedge T_2 = \bigwedge T_3$;

– if $T_2 = E_1$, then this is symmetric;

– if $T_1 = T_2 = E_{01}$, then $T_3 = T_1$, hence $\bigwedge T_1 \,\&\, \bigwedge T_2 = \bigwedge T_3$.

Lemma 5.4.6. *If* (i)–(viii) *hold for* DIL$\leqslant H'$ *and* DIL$\leqslant H''$, *then they hold for* DIL$\leqslant(H' + H'')$. (*This lemma, combined with Case* 5.4.5 *above, permits to treat the case of kind* 1: *if* (i)–(viii) *hold for* DIL$\leqslant H'$, *they they hold for* DIL$\leqslant(H' + \underline{1})$.)

Proof. Write $H = H' + H''$. First we extend $\bigwedge$ to DIL$\leqslant H$:

– if H_1 is a predecessor of H, then, either H_1 is a predecessor of H', or $H_1 = H' + H_1''$, where H_1'' is a predecessor of H''. So it is possible to write in both cases $H_1 = H_1' + H_1''$ with H_1' (resp. H_1'') equal to or predecessor of H' (resp. H''); this is still true when $H_1 = H$.

– if F is an object of DIL$\leqslant H$, then let H_1 (equal to H or predecessor of H) be such that $T \in I(F, H_1)$ for some T. The decomposition $H_1 = H_1' + H_1''$ induces a decomposition $F = F' + F''$, $T = T' + T''$ $T' \in I(F', H_1')$, $T'' \in I(F'', H_1'')$. Let x, y, y', y'', f, be the respective lengths of F, H_1, H_1', H_1'', T; then write $f = f' + f''$, with $f' \in I(x', y')$, $f'' \in I(x'', y'')$. If $F = \sum_{i<x} F_i$, if $T = \sum_{i<f} T_i$, then let

$$F' = \sum_{i<x'} F_i, \qquad F'' = \sum_{i<x''} F_{x'+i},$$

$$T' = \sum_{i<f'} T_i, \qquad T'' = \sum_{i<f''} T_{x'+i},$$

so $T' \in I(F', H_1')$, $T'' \in I(F'', H_1'')$.

We define $\bigwedge F = \bigwedge F' \circ_s \bigwedge F''$. This is possible, since F' (resp. F'') is an object of DIL$\leqslant H'$ (resp. DIL$\leqslant H''$). If $F = G' + G''$ is a similar decomposition, then, for instance LH$(G')<$LH(F'), so write $F' = G' + G_1'$ (with $G_1' = \sum_{i<x'-\mathrm{LH}(G')} F_{\mathrm{LH}(G')+i}$). So $G'' = G_1' + F''$: then

$$\bigwedge F = \bigwedge F' \circ_s \bigwedge F'' = \bigwedge G' \circ_s \bigwedge G_1' \circ_s \bigwedge F'' = \bigwedge G' \circ_s \bigwedge G''$$

(we have used the associativity of $\circ_s$, and the property (v) with $F' = G' + G_1'$ in DIL$\leqslant H'$, with $G'' = G_1' + F''$ in DIL$\leqslant H''$).

If $T\in I(F,G)$, then, given a decomposition $G=G'+G''$ (with G' in DIL$\leqslant H'$, G'' in DIL$\leqslant H''$), then it is possible to write $T=T'+T''$, $T'\in I(F',G')$, $T''\in I(F'',G'')$. This is done exactly as above, replacing H_1' by G', H_1'' by G''; then define $\bigwedge T=\bigwedge T' \circ_s \bigwedge T''$. This definition is independent of the given decomposition of G. If $G=D'+D''$, inducing a decomposition $T=U'+U''$, then, for instance $h(D')\leqslant h(G')$, so write $G'=D'+D_1'$, and so $T'=U'+U_1'$; then:

$$\bigwedge T=\bigwedge T'\circ_s \bigwedge T''=\bigwedge U'\circ_s \bigwedge U_1\circ_s \bigwedge T''=\bigwedge U'\circ_s \bigwedge U''.$$

(i): We prove that $\bigwedge(TU)=(\bigwedge T)(\bigwedge U)$: given a decomposition $G=G'+G''$ (G' in DIL$\leqslant H'$, G'' in DIL$\leqslant H''$), then $T\in I(F,G)$ is decomposed in $T'+T''$, so $F=F'+F''$; the decomposition $F=F'+F''$ induces a decomposition $U=U'+U''$; furthermore, $TU=T'U'+T''U''$, hence:

$$\begin{aligned}\bigwedge(TU)&=\bigwedge(T'U')\circ_s \bigwedge(T''U'')=(\bigwedge T')(\bigwedge U')\circ_s(\bigwedge T'')(\bigwedge U'')\\&=(\bigwedge T'\circ_s \bigwedge U')(\bigwedge T''\circ_s \bigwedge U'')=(\bigwedge T)(\bigwedge U).\end{aligned}$$

(ii) and (iii): Immediate.

(iv): If G, perfect $\neq \underline{1}$, is in DIL$\leqslant H$, then write $G=G'=G''$, with G' in DIL$\leqslant H'$, G'' in DIL$\leqslant H''$, then $G=G'$ or $G=G''$, so the property is already in the hypothesis.

(v): If $G=G'+G''$, then write $G=F'+F''$, with F' in DIL$\leqslant H'$, F'' in DIL$\leqslant H''$, then two subcases:

–if LH$(G')\leqslant$LH(F'), write $F'=G'+G_1'$, so $G''=G_1'+F''$, and

$$\bigwedge G=\bigwedge F'\circ_s \bigwedge F''=\bigwedge G'\circ_s \bigwedge G_1'\circ_s \bigwedge F''=\bigwedge G'\circ_s \bigwedge G''.$$

–if LH$(G')\geqslant$LH(F'), write $G'=F'+G_1'$, so $F''=G_1'+G''$, and

$$\bigwedge G=\bigwedge F'\circ_s \bigwedge F''=\bigwedge F'\circ_s \bigwedge G_1'\circ_s \bigwedge G''=\bigwedge G'\circ_s \bigwedge G''.$$

The property $\bigwedge(T'+T'')=\bigwedge T'\circ_s \bigwedge T''$ is obtained in the same way.

(vi): If $F=F'+F''$, with F' in DIL$\leqslant H'$, F'' in DIL$\leqslant H''$, then $E_{0F}=E_{0F'}+E_{0F''}$, so

$$\begin{aligned}(\bigwedge E_{0F})(x,y)(z)&=(\bigwedge E_{0F'})(E_x,(\bigwedge E_{0F''})(x,y))(z)\\&=(\bigwedge F')(x,(\bigwedge E_{0F''})(x,y)(z))=(\bigwedge F')(x,(\bigwedge F'')(x,z))\\&=(\bigwedge F)(x,z)\end{aligned}$$

(we have used property (vi) for F' (under the form $(\bigwedge E_{0F'})(E_x,f)(z)=(\bigwedge F')(x,f(z))$) and for F'').

(vii): If $(F,T_i)=\varinjlim(F_i,T_{ij})$, then let $F=F'+F''$, F' in DIL$\leqslant H'$, F'' in DIL$\leqslant H''$, then it is possible to write $T_i=T_i'+T_i''$, $F_i=F_i'+F_i''$, $T_{ij}=T_{ij}'+T_{ij}''$, and obviously:

$$(F',T_i')=\varinjlim(F_i',T_{ij}')\quad\text{and}\quad(F'',T_i'')=\varinjlim(F_i'',T_{ij}),$$

then

$$\begin{aligned}(\bigwedge F,\bigwedge T_i)&=(\bigwedge F'\circ_s \bigwedge F'',\bigwedge T_i'\circ_s \bigwedge T_i'')\\&=\varinjlim(\bigwedge F_i'\circ_s \bigwedge F_i'',\bigwedge T_{ij}'\circ_s \bigwedge T_{ij}'')=\varinjlim(\bigwedge F_i,\bigwedge T_{ij}).\end{aligned}$$

(We have used (vi) for (F_i', T_{ij}) and (F_i'', T_{ij}) and the fact that $\circ_s$ commutes to $\varinjlim$.)

(viii): If $T_i \in I(F_i, G)$, with $T_1 \,\&\, T_2 = T_3$, then write $G = G' + G''$, with G' in DIL$\leqslant H'$, G'' in DIL$\leqslant H''$, then this decomposition induces a decomposition $T_i = T_i' + T_i''$, and it is immediate that $T_3' = T_1' \,\&\, T_2'$, $T_3'' = T_1'' \,\&\, T_2''$. Then

$$\bigwedge T_3 = \bigwedge T_3' \circ_s \bigwedge T_3'' = (\bigwedge T_1' \,\&\, \bigwedge T_2') \circ_s (\bigwedge T_1'' \,\&\, \bigwedge T_2'') = \bigwedge T_1 \,\&\, \bigwedge T_2.$$

(We have used the fact that (vii) holds for the T_i' and the T_i'', and the fact that $\circ_s$ commutes to pull-backs.)

Case 5.4.7. If H is of kind ω; assume that (i)–(viii) hold for DIL$<H$; we shall define $\bigwedge G$ and $\bigwedge T$ when $h(G) = h(T) = H$. Assume that $G = \sum_{i<y} G_i'$ is the decomposition of G as a sum of perfect dilators, then y is limit: if $T \in I(G, H)$, then consider the function $\mathrm{LH}(T) = f$, from y to $z = \mathrm{LH}(H)$. If $\hat{f}(y) \neq z$, then it is possible to write $H = H' + H''$, $T = T' + E_{0H''}$, $T' \in I(G, H')$. Let (if $H = \sum_{i<z} H_i$), $H' = \sum_{i<\hat{f}(y)} H_i, \ldots$. Observe that $H'' \neq \underline{0}$, so H' is a predecessor of H, and $h(G) \leqslant h(H') < H$, contrarily to the assumptions. So $\hat{f}(y) = z$, hence y is limit. This shows that G is of kind ω.

Let $G_i = \sum_{j<i} G_j'$ then $(G, E_{G_iG}) = \varinjlim(G_i, E_{G_iG_j})$ (Example 4.1.5(iii)). Define $\bigwedge G = \varinjlim^*(\bigwedge G_i, \bigwedge E_{G_iG_j})$. This direct limit exists in DIL because of Lemma 5.3.2. Let $F_i = \bigwedge G_i$, $T_{ij} = \bigwedge E_{G_iG_j}$, $F_{ij} = \bigwedge(\sum_{k<j-i} G_{i+k}')$, then the conditions of the lemma are fulfilled because of the conditions (v) and (vi) in the induction hypothesis.

By the way, observe that the formula above is valid for an arbitrary G (if $h(G) < H$, this is the induction hypothesis (vii)). If $T \in I(F, G)$, with $h(T) = h(G) = H$, then (with obvious notations) $\bigwedge F = \varinjlim_x^*(\bigwedge F_i, \bigwedge E_{F_iF_j})$. If $T_i' \in I(F_i', G_{f(i)}')$ are such that $T = \sum_{i<x} T_i'$, then define $T_i \in I(F_i, G_i)$ by $T_i = \sum_{j<i} T_j'$, then $T = \varinjlim(T_i)$, and let $\bigwedge T = \varinjlim(\bigwedge T_i)$. By the way, note that $(\bigwedge G, \bigwedge E_{G_iG}) = \varinjlim(\bigwedge G_i, \bigwedge E_{G_iG_j})$.

(i): $\bigwedge(TU) = (\bigwedge T)(\bigwedge U)$. Write as above $T = \varinjlim(T_i)$, $U = \varinjlim(U_i)$, $TU = \varinjlim(V_i)$, then it is immediate that $V_i = T_{g(i)} U_i$, with $g = \mathrm{LH}(U)$; then

$$\begin{aligned}\bigwedge(TU) &= \varinjlim(\bigwedge T_{g(i)} U_i) = \varinjlim(\bigwedge T_{g(i)}, \bigwedge U_i)\\ &= \varinjlim(\bigwedge T_i)\varinjlim(\bigwedge U_i) = (\bigwedge T)(\bigwedge U).\end{aligned}$$

(We have used the induction hypothesis (i) for $T_{g(i)} U_i$ and Proposition 1.3.7(ii).)

(ii)–(iv): Trivial.

(v): Assume that $T' \in I(F', G')$, $T'' \in I(F'', G'')$, let $T = T' + T''$, $G = G' + G''$; two subcases:

–if $G'' = \underline{0}$, then

$$\bigwedge G = \bigwedge G' = \bigwedge G' \circ_s \mathrm{Id} = \bigwedge G' \circ_s \bigwedge G''$$

and

$$\bigwedge T = \bigwedge T' = \bigwedge T' \circ_s \mathrm{Id} = \bigwedge T' \circ_s \bigwedge T''$$

–if $G'' \neq \underline{0}$, let $x, x', x'', y, y', y'', f, f', f''$ be the respective lengths of $F, F', F'', G, G', G'', T, T', T''$. Observe that $x = x' + x''$, $y = y' + y''$, $f = f' + f''$; define F_i, F_i'',

G_1, G''_i, T_i, T''_i as above, and observe that $F_{x+i} = F' + F''_i$, $G_{g+i} = G' + G''_i$, $T_{x+i} = T' + T''_i$, then

$$\begin{aligned}
\bigwedge G &= \varinjlim_{y}{}^{*}(\bigwedge G_i, \bigwedge E_{G_iG_j}) = \varinjlim_{y''}{}^{*}(\bigwedge(G' + G''_i), \bigwedge(E_{G'} + E_{G''_iG''_j})) \\
&= \varinjlim_{y''}{}^{*}(\bigwedge G' \circ_s \bigwedge G''_i, \bigwedge E_{G'} \circ_s \bigwedge E_{G''_iG''_j}) \\
&= \bigwedge G' \circ_s \varinjlim_{y''}{}^{*}(\bigwedge G''_i, \bigwedge E_{G''_iG''_j}) = \bigwedge G' \circ_s \bigwedge G''.
\end{aligned}$$

$$\begin{aligned}
\bigwedge T &= \varinjlim_{x}(\bigwedge T_i) = \varinjlim_{x''}(\bigwedge(T' + T''_i) \\
&= \varinjlim_{x''}(\bigwedge T' \circ_s \bigwedge T''_i) = \bigwedge T' \circ_s \varinjlim_{x''}(\bigwedge T''_i = \bigwedge T' \circ_s \bigwedge T''.
\end{aligned}$$

(We have used the induction hypothesis (v) for G', T' (observe that $h(G') < H$) and for G''_i, T''_i, together with the fact that $\circ_s$ commutes to direct limits.)

(vi): $(\bigwedge G, \bigwedge E_{G_iG}) = \varinjlim(\bigwedge G_i, \bigwedge E_{G_iG_j})$, so apply Lemma 5.3.2(ii) with $F_i = \bigwedge G_i$, $T_{ij} = \bigwedge E_{G_iG_j}$, $F = \bigwedge G$, $T_i = \bigwedge E_{G_iG}$: (observe that $G_0 = \underline{0}$, so $\bigwedge G_0 = \mathrm{Id}$)

$$(\bigwedge E_{0G})(x, y)(z) = (\bigwedge E_{G_0G})(x, y)(\bigwedge G_0)(x, z) = (\bigwedge G)(x, z).$$

(vii) and (viii): Let $\mathrm{SFD} \leqslant H = \mathrm{SFD} \cap (\mathrm{DIL} \leqslant H) = \mathrm{SFD} \cap (\mathrm{DIL} < H)$, then the functor $\bigwedge \restriction (\mathrm{DIL} < H)$ commutes to $\varinjlim$ and &, so it coincides with the extension by direct limits of $\bigwedge \restriction (\mathrm{SFD} \leqslant H)$ to $\mathrm{DIL} < H$. $\bigwedge \restriction (\mathrm{SFD} \leqslant H)$ can be extended to $\mathrm{DIL} \leqslant H$ (but takes its values eventually in PIL), in such a way that it commutes to direct limits This extension is unique up to isomorphism, and enjoys $\bigwedge G = \varinjlim{}^{*}((\bigwedge G_i, \bigwedge E_{G_iG_j})$, $\bigwedge T = \varinjlim(\bigwedge T_i)$; this extension coincides therefore with the extension constructed above. But the extension of $\bigwedge \restriction (\mathrm{SFD} \leqslant H)$ by direct limits to $\mathrm{DIL} \leqslant H$ commutes to direct limits and pull-backs (Remark 4.4.6(iii)).

Case 5.4.8. If H is of kind Ω, then, by induction hypothesis, (i)–(viii) hold in $\mathrm{DIL} < H$. It is enough to treat the case H perfect, because, if H is not perfect, then write $H = H' + H''$, then $\mathrm{DIL} \leqslant H'$ and $\mathrm{DIL} < H''$ are subcategories of $\mathrm{DIL} < H$, and by the case H'' perfect. (i)–(viii) hold in $\mathrm{DIL} \leqslant H'$ and $\mathrm{DIL} \leqslant H''$, hence in $\mathrm{DIL} \leqslant H$, by Lemma 5.4.6.

We define $\bigwedge G$ and $\bigwedge T$ when $h(G) = h(T) = H$. If $U \in I(G, H)$, then either $G = \underline{0}$ (absurd) or G is perfect (because $\mathrm{LH}(G)$ is smaller than $\mathrm{LH}(H) = 1$) and $G \neq 1$, because $G(0) \leqslant H(0) = 0$. So G is of kind Ω, and one can define $\bigwedge G$ by means of (iv).

If $T \in I(F, G)$, then either $F = \underline{0}$, and $\bigwedge T$ is defined by means of (vi), or F is perfect $\neq \underline{1}$, and $T \in \Omega I(F, G)$, so define $\bigwedge T$ by (iv).

(i): It is immediate that $\bigwedge G$ defined by means of (iv) is a functor from ON^2 to ON. Furthermore $\bigwedge G$ commutes to $\varinjlim$ and &:

$$(\bigwedge G)(\cdot, y) = \prod_{t<y} (\underline{1} + \bigwedge_t G)(\cdot, 0),$$

it is immediate that $(\bigwedge G)(\cdot, y)$ commutes to $\underrightarrow{\lim}$ and &. So it suffices to prove that $(\bigwedge G)(x, \cdot)$ commutes to $\underrightarrow{\lim}$ and &.

If $(y, g_i) = \underrightarrow{\lim}(y_i, g_{ij})$; if $\iota < y$, let $I_\iota = \{i; \iota \in \mathrm{rg}(g_i)\}$. Define $(F^\iota, T_i^\iota) = (F_i^\iota, T_{ij}^\iota) = ({}_\iota G, E_{{}_\iota G})$, and apply Proposition 5.3.8(i):

$$\left(\prod_{\iota<y}(\underline{1}+\bigwedge_\iota G), \prod_{\iota<g_i}(E_1+\bigwedge E_{{}_\iota G})\right) = \underrightarrow{\lim}_I\left(\prod_{\iota<y_i}(\underline{1}+\bigwedge_\iota G), \prod_{\iota<g_{ij}}(E_1+\bigwedge E_{{}_\iota G})\right).$$

Applying both sides to the pair $(x, 0)$, one gets:

$$((\bigwedge G)(x, y), (\bigwedge G)(E_x, g_i)) = \underrightarrow{\lim}((\bigwedge G)(x, y_i), (\bigwedge G)(E_x, g_{ij}))$$

If $g_i \in I(y_i, y)$ $(i = 1, 2, 3)$ and $g_3 = g_1 \,\&\, g_2$, then Proposition 5.3.8(ii) yields:

$$\prod_{\iota<g_3}(E_1+\bigwedge E_{{}_\iota G}) = \left(\prod_{\iota<g_1}(E_1+\bigwedge E_{{}_\iota G})\right) \& \left(\prod_{\iota<g_2}(E_1+\bigwedge E_{{}_\iota G})\right)$$

and applying both sides to the pair (E_x, E_0), one gets:

$$(\bigwedge G)(E_x, g_3) = (\bigwedge G)(E_x, g_1) \,\&\, (\bigwedge G)(E_x, g_2).$$

We show that the functor $(\bigwedge G)(x, \cdot)$ enjoys (FL). If $y \leq y'$

$$\begin{aligned}
(\bigwedge G)(E_x, E_{yy'}) &= \prod_{\iota<E_{yy'}}(E_1+\bigwedge E_{{}_\iota G})(E_x, E_0) \\
&= \left(\prod_{\iota<E_y}(E_1+\bigwedge E_{{}_\iota G}) \circ_s \prod_{\iota<E_{0y'}}(E_1+\bigwedge E_{{}_\iota G})\right)(E_x, E_0) \\
&= (E_F \circ_s \mu_K)(E_x, E_0),
\end{aligned}$$

with $F = \prod_{\iota<y}(\underline{1}+\bigwedge_{{}_\iota G})$, $K = \prod_{\iota<y'-y}(\underline{1}+\bigwedge_{{}_\iota G})$, but

$$\begin{aligned}
(E_F \circ_s \mu_K)(x, 0) &= F(E_x, \mu_K(x, 0)) = F(E_x, E_{0K(x,0)}) \\
&= E_{F(x,0)F(x,K(x,0))} = E_{(\bigwedge G)(x,y)(\bigwedge G)(x,y')}.
\end{aligned}$$

It is immediate that, if $T \in I(F, G)$, then $\bigwedge T \in I(\bigwedge F, \bigwedge G)$. We show that $\bigwedge(TU) = (\bigwedge T)(\bigwedge U)$, when $h(T) = h(G) = h(H)$, $T \in I(F, G)$:

– If $F \neq \underline{0}$, and $U \neq E_{\underline{0}F}$, then

$$\begin{aligned}
(\bigwedge TU)(x, y) &= \prod_{\iota<E_y}(E_1+\bigwedge_\iota(TU))(x, 0) = \prod_{\iota<E_y}(E_1+\bigwedge_\iota T_\iota U)(x, 0) \\
&= \left(\left(\prod_{\iota<E_y}(E_1+\bigwedge_\iota T)\right)\left(\prod_{\iota<E_y}(E_1+\bigwedge_\iota U)\right)(x, 0)\right) \\
&= \left(\left(\prod_{\iota<E_y}(E_1+\bigwedge_\iota T)\right)(x, 0)\right)\left(\left(\prod_{\iota<E_y}(E_1+\bigwedge_\iota U)\right)(x, 0)\right) \\
&= (\bigwedge T(x, y))(\bigwedge U)(x, y);
\end{aligned}$$

– in general, observe that, from the obvious equalities: $\prod_{\iota<y}(\underline{1}+\mathrm{Id})(x, z) = y + z$, $\prod_{\iota<g}(\underline{1}+\mathrm{Id})(f, h) = g + h$, Definition 5.4.1(iii) also holds when $G = \underline{0}$ (with ${}_\iota G = \underline{0}$, ${}_{g\iota}G = E_{\underline{0}}$), and when $T = E_{\underline{0}G}$, with $G = \underline{0}$ or G perfect (with ${}_\iota T = E_{\underline{0}{}_\iota G}$). From this one can prove $(\bigwedge TU) = (\bigwedge T)(\bigwedge U)$ in general on the model of the case above.

(ii)–(iv): Trivial.

(v); Observe that, if $h(F'+F'')=h(H)$, then one of F' and F'' is $\underset{\sim}{0}$, and we conclude as in Case 5.4.5(v) similarily for $T'+T''$.

(vi): Trivial by construction, if $h(F)=H$, but we have to verify the possibility of defining $\bigwedge E_{0F}$ by $\mu_{\bigwedge F}$, i.e., that $\bigwedge F$ is regular. First $(\bigwedge F)(x, y+1) = K(x, 1+L(x, y))$, with $K=\prod_{t<y}(\underset{\sim}{1}+\bigwedge_t F)$, $L=\bigwedge_y F$, whereas $(\bigwedge F)(x, y)=K(x, y)$, so $K(x, y)<K(x, y+1)$.

If $f\in I(x, x')$, if $g\in I(z, z')$, then we show that

$$(\bigwedge F)(f, g+E_1)((\bigwedge F)(x, z))=(\bigwedge F)(x', z').$$

This will establish regularity. Let $T=\prod_{t<g}(E_1+\bigwedge_{gt}F)$, and let $U=\bigwedge_{(g+E_1)z}F$, then, obviously,

$$(\bigwedge F)(f, g+E_1)=(T\circ_s(E_1+U))(f, 0)=T(f, E_1+U(f, E_0))$$

and let

$$G=\prod_{t<z}(\underset{\sim}{1}+\bigwedge_t F), \qquad G'=\prod_{t<z}(\underset{\sim}{1}+\bigwedge_t F),$$

so $T\in I_r(G, G')$, hence

$$(\bigwedge F)(f, g+E_1)(G(x, 0) = T(f, E_1+U(f, E_0))(G(x, 0))=G(x', 0).$$

It suffices now to observe that $(\bigwedge F)(x, z)=G(x, 0)$, whereas $(\bigwedge F)(x, z')=G'(x, 0)$.

(vii): If $(F, T_i)=\varinjlim_I(F_i, T_{ij})$, with $h(F)=H$, then one can assume, by restricting I to a cofinal subset, that $F_i\neq\underset{\sim}{0}$ for all i. So we have a direct system in ΩDIL, with its limit in Ω_{DIL}: Proposition 5.3.8(i) yields

$$\left(\prod_{t<y}(\underset{\sim}{1}+\bigwedge_t F), \prod_{t<E_y}(E_1+\bigwedge_t T_i)\right)=\varinjlim\left(\prod_{t<y}(\underset{\sim}{1}+\bigwedge_t F_i), \prod_{t<E_y}(E_1+\bigwedge_t T_{ij})\right)$$

(using: $(y, E_y)=\varinjlim(y, E_y)$ and $({}_tF, {}_tT_i)=\varinjlim({}_tF_i, {}_tT_{ij})$, together with the induction hypothesis (vii)).

If one applies $(x, 0)$ to both sides, one gets

$$((\bigwedge F)(x, y), (\bigwedge T_i)(x, y))=\varinjlim((\bigwedge F_i)(x, y), (\bigwedge T_{ij})(x, y)).$$

which implies $(\bigwedge F, \bigwedge T_i)=\varinjlim(\bigwedge F_i, \bigwedge T_{ij})$.

(viii): Assume that $h(G)=H$, and that $T_i\in I(F_i, G)$ $(i=1,2,3)$, with $T_3=T_1\ \&\ T_2$. We have already observed, in the proof of (i), that

$$T_i(x, y)=\prod_{t<E_y}(E_1+\bigwedge_t T_i)(x, 0)$$

is true, even when $F_i=\underset{\sim}{0}$, with in that case ${}_tT_i=E_{0,G}$. Observe that ${}_tT_3={}_tT_1\ \&\ {}_tT_2$, so apply Proposition 5.3.8(ii), in order to get

$$\left(\prod_{t<E_y}(E_1+\bigwedge_t T_3)\right)=\left(\prod_{t<E_y}(E_1+\bigwedge_t T_1)\right)\&\left(\prod_{t<E_y}(E_1+\bigwedge_t T_2)\right)$$

(using the induction hypothesis (viii) which gives $\bigwedge_t T_3 = \bigwedge_t T_1 \,\&\, \bigwedge_t T_2$, and $E_y = E_y \,\&\, E_y$). If one applies $(x, 0)$ to both sides, one gets

$$(\bigwedge T_3)(x, y) = (\bigwedge T_1)(x, y) \,\&\, (\bigwedge T_2)(x, y),$$

which implies

$$\bigwedge T_3 = \bigwedge T_1 \,\&\, \bigwedge T_2.$$

Remarks 5.4.9. (i) The products

$$\prod_{t<y} (\underline{1} + \bigwedge_t F), \qquad \prod_{t<g} (E_1 + \bigwedge_{gt} F) \quad \text{and} \quad \prod_{t<E_y} (E_1 + \bigwedge_t T),$$

which are essential in the case of perfect dilators $\neq \underline{1}$, are very closely connected to the functor $\bigwedge\mathrm{SEP}(F)(\cdot, y)$ and the morphisms $\bigwedge\mathrm{SEP}(F)(\cdot, g)$, $\bigwedge\mathrm{SEP}(T)(\cdot, y)$. An application of Definition 5.4.1(i), using the fact that $\mathrm{SEP}(F)(\cdot, y) = \sum_{t<y} {}_tF$, etc. . . , shows that

$$\bigwedge\mathrm{SEP}(F)(\cdot, y) = \prod_{t<y} \bigwedge_t F, \qquad \bigwedge\mathrm{SEP}(F)(\cdot, g) = \prod_{t<g} \bigwedge_{gt} F,$$

$$\bigwedge\mathrm{SEP}(T)(\cdot, y) = \prod_{t<E_y} \bigwedge_t T,$$

so the only difference lies in the additional terms $\underline{1}$ and E_1 used in these products. They are needed in order to ensure regularity, and also commutation to & in the case $T_1 \,\&\, T_2 = E_{0G}$, with G perfect $\neq \underline{1}$. But, if one forgets this rather technical motivation, the idea, in the case F perfect $\neq \underline{1}$, is to write:

$$\bigwedge F(x, y) = (\bigwedge\mathrm{SEP}(F)(\cdot, y))(x, 0), \qquad \bigwedge F(f, g) = (\bigwedge\mathrm{SEP}(F)(\cdot, g))(f, E_0),$$

$$\bigwedge T(x, y) = (\bigwedge\mathrm{SEP}(T)(\cdot, y))(x, 0).$$

Unfortunately, this definition would lead us out of regular bilators, so we are forced to the smeall change which consists in adding the $\underline{1}$ and E_1 in the products.

(ii) By the way, note that $\bigwedge$ transforms $\sum$ into $\prod$ for arbitrary objects and morphisms, not only perfect (immediate).

Proposition 5.4.10. *Assume that F, G are dilators, G perfect $\neq \underline{1}$ or $G = \underline{0}$, and that $T \in I(F, G)$, then*

(i) $(\bigwedge G) \circ_s (\underline{1} + \mathrm{Id}) = \underline{1} + \bigwedge(G \circ (\underline{1} + \mathrm{Id}))$;

(ii) $(\bigwedge T) \circ_s (E_1 + E_{\mathrm{Id}}) = E_1 + \bigwedge(T \circ (E_1 + E_{\mathrm{Id}}))$.

Proof. (i) Let $G' = G \circ (\underline{1} + \mathrm{Id})$, with $G \neq \underline{0}$ ($G = \underline{0}$ is trivial), then $G' = {}_0G + G''$, with G'' perfect, and ${}_tG'' = {}_{1+t}G$, ${}_{gt}G'' = {}_{(E_1+g)(1+t)}G$; so

$$(\bigwedge G)(x, 1+y) = \prod_{t<1+y} (\underline{1} + \bigwedge_t G)(x, 0) = \Big((\underline{1} + \bigwedge_0 G) \circ \prod_{t<y} (\underline{1} + \bigwedge_{1+t} G)\Big)(x, 0)$$

$$= 1 + \bigwedge_0 G(x, \bigwedge G''(x, y)) = \bigwedge G'(x, y).$$

For the same reason, $(\bigwedge_0 G)(f, E_1 + g) = \bigwedge_0 G(f, \bigwedge G''(f, g)) = \bigwedge G'(f, g)$.

(ii) This is obtained by the same method.

6. Dendroids

6.1. Dendroids

Definition 6.1.1. A *dendroid of type* x is a pair (x, D), where x is an ordinal, and D is a set of sequences $s = (a_0, \ldots, a_p)$ of ordinals, such that:

(i) if $s = (a_0, \ldots, a_p) \in D$, then p is even,

(ii) if $s = (a_0, \ldots, a_p) \in D$ and $q < p$, then $(a_0, \ldots, a_q) \notin D$.

In order to state the remaining properties, we define D^*: $s \in D^*$ iff $s = (\ )$ or $s * s' \in D$ for some s'. ($*$ denotes concatenation).

(iii) if $s = (a_0, \ldots, a_{2p}) \in D^*$, if $s*(t)$, $s*(u) \in D^*$ and $t \leq u$, then $a_{2i+1} < t \leq u$ or $t \leq u < a_{2i+1}$ for all $i < p$. (As a consequence $a_1, a_3, \ldots$ are pairwise distinct.) Moreover, $a_{2i+1} < x$.

(iv) If $s = (a_0, \ldots, a_{2p-1}) \in D^*$, then the set $\{a; s*(a) \in D^*\}$ is an ordinal, i.e., $(a_0, \ldots, a_{2p-1}, b) \in D^*$ and $a < b \rightarrow (a_0, \ldots, a_{2p-1}, a) \in D^*$.

(v) There is no sequence (a_n) such that, for all n, $(a_0, \ldots, a_{n-1}) \in D^*$.

Remarks 6.1.2. (i) We shall usually, when the context is clear, identify a dendroid (x, D) with the second component D.

(ii) Condition (v) is a well-foundedness condition. The tree D^* has no strictly decreasing sequence for the Brouwer–Kleene ordering.

(iii) The concept of dendroid is reminiscent of the concept of ordinal tree, due to Jervell [11], however, there are essential differences and the greek-derived 'dendroid' permits to avoid confusions. In this section, we try to parallel Jervell's terminology, when possible (for instance, homogeneous, strongly homogeneous dendroids).

Definition 6.1.3. If (x, D) and (x', D') are dendroids, then $I(x, D; x', D')$ consists of all pairs (f, g), with:

(i) $f \in I(x, x')$,

(ii) g is a function from D^* to D'^*, which sends D into D',

(iii) $g((\ )) = (\ )$,

(iv) $g(s*(a)) = g(s)*(f(a))$, if $s = (a_0, \ldots, a_{2p})$,

(v) $g(s*(a)) = g(s)*(b)$ for some b (depending on s, a), otherwise.

Furthermore, if $g(s*(a')) = g(s)*(b')$ and $a < a'$, then $b < b'$.

Remark 6.1.4. If $(f, g) \in I(x, D; x', D')$, if $(f', g') \in I(x', D'; x'', D'')$, then $(f'f, g'g) \in I(x, D; x'', D'')$.

Example 6.1.5. Given a dendroid (y, D') and $f \in I(x, y)$, we shall define a new dendroid $(x, D) = f^{-1}(y, D')$ (we shall also write $D = f^{-1}(D')$) by:

(i) In D', remove all sequences $(a_0, \ldots, a_{2p})$, with $a_{2i+1} \notin \mathrm{rg}(f)$ for some $i < p$. One obtains a set D'' of sequences enjoying all properties of dendroids except perhaps (iv). We define D'''^* as in Definition 6.1.1. The process of construction of D'' and D'''^* is called a *mutilation*. We denote D'' by ${}^fD$.

(ii) There exists one and only one dendroid D and one and only one function g from D^* to D'^* such that $(f, g) \in I(x, D; y, D')$ and $\mathrm{rg}(g) = D'''^*$. We define the members s of D^* and $g(s)$ by induction on n such that $s = (a_0, \ldots, a_{n-1})$: if $n = 0$, then $s \in D^*$, and $g(s) = s$; if $n \neq 0$, write $s = t * (a)$, and distinguish two subcases:

(1) if n is even, then $s \in D^*$ iff $t \in D^*$ and $g(t) * (f(a)) \in D'''^*$. Define $g(s) = g(t) * (f(a))$;

(2) if n is odd, then $s \in D^*$ iff $t \in D^*$ and the order type of the set $X = \{u; g(t) * (u) \in D'''^*\}$ is strictly greater than a. Define $g(s) = g(t) * (b)$, where b is the ath element of X.

(iii) The function g defined above will be called the *mutilation function* and is denoted by $m_f^{D'}$ (or m_f). So $(f, m_f) \in I(x, f^{-1}(D'); y, D')$.

Definition 6.1.6. The category DEN of dendroids is defined by the following data:

objects: dendroids (x, D);

morphisms from (x, D) *to* (x', D'): the elements of $I(x, D; x', D')$.

Remark 6.1.7. In DEN, there are only trivial isomorphisms. If $(f, f') \in I(x, D; x', D')$, $(g, g') \in I(x', D'; x, D)$ enjoy $(fg, f'g') = (E_{x'}, \mathrm{id}_{D'^*})$, $(gf, g'f') = (E_x, \mathrm{id}_{D^*})$, then $x = x'$, and $f = g = E_x$. Also, it is immediate that condition (iv) of dendroids implies $D = D'$ and $f' = g' = \mathrm{id}_{D^*}$.

6.2. *The functors type and height*

Definition 6.2.1. The functor *type* from DEN to ON is defined by:

$$\underline{t}(x, D) = x, \qquad \underline{t}(f, g) = f.$$

Proposition 6.2.2. *In DEN,* $(x, D; f_i, g_1) = \varinjlim(x_i, D_i; f_{ij}, g_{ij})$ *iff Definition* 1.3.3(i)–(iii) *holds and*

(i) $x = \bigcup_i \mathrm{rg}(f_i)$ (*i.e.*, $(x, f_i) = \varinjlim(x_i, f_{ij})$);

(ii) *every point in D is in the range of some* g_i: *hence* $D^* = \bigcup_i \mathrm{rg}(g_i)$.

Proof. (1) Assume that $(x, D; f_i, g_i) = \varinjlim(x_i, D_i; f_{ij}, g_{ij})$. Define $X = \bigcup_i \mathrm{rg}(g_i)$. $\Delta = \bigcup_i \mathrm{rg}(g_i)$. Define x' and $h \in I(x', x)$ by $\mathrm{rg}(h) = X$, and D' by $s \in D'^* \Leftrightarrow m_h(s) \in \Delta$. It is immediate that D' is a dendroid. Define a function k from D'^* to D^* by $k(s) = m_h(s)$. One can define $(f_i', g_i') \in I(x_i, D_i; x', D')$ by $f_i = hf_i'$, $g_i = kg_i'$. $(x', D'; kf_i', g_i')$ enjoys conditions 1.3.3(i)–(iii), and by Definition 1.3.3(iv) applied to $(x, D; f_i, g_i)$, there is an unique $(h', k') \in I(x, D; x', D')$ such that $f_i' = h'f_i$, $g_i' = k'g_i$ for all $i \in I$. From this $f_i = hh'f_i$, $g_i = kk'g_i$, and condition 1.3.3(iv) applied again yields (unicity) $(hh', kk') = \text{identity}$, so h and k are surjective, i.e., $X = x$, $\Delta = D^*$.

(2) Assume that (i) and (ii) hold. We show that Definition 1.3.3(iv) is true. If $(x', D'; f_i', g_i')$ is any other solution of Definition 1.3.3(i)–(iii), then define $h \in I(x, x')$ by: $h(f_i(z)) = f_i'(z)$, and k from D^* to D' by $k(g_i(s)) = g_i'(s)$. This is possible because of (i) and (ii), then $(h, k)(f_i, g_i) = (f_i', g_i')$ for all $i \in I$, and this is obviously the only solution.

Corollary 6.2.3. (i) *The functor* $\underline{t}$ *commutes to direct limits;*

(ii) *in* DEN, $(x, D; f_i, m_{f_i}) = \varinjlim(x_i, D_i; f_{ij}, m_{f_{ij}})$ *iff conditions* 1.3.3(i)–(iii) *are satisfied, and* $(x, f_i) = \varinjlim(x_i, f_{ij})$.

Proof. (i) Immediate from Proposition 6.2.2.

(ii) Assume Definition 1.3.3(i)–(iii) and $(x, f_i) = \varinjlim(x_i, f_{ij})$. Given $s \in D$, $s = (\ldots, a_{2k}, a_{2k+1}, \ldots)$ choose i such that $a_1, a_3, a_5, \ldots \in \mathrm{rg}(f_i)$. This is possible by hypothesis, then $s \in \mathrm{rg}(m_{f_i})$. We have seen that conditions (i) and (ii) of Proposition 6.2.2 are true, so $(x, D; f_i, m_{f_i}) = \varinjlim(x_i, D_i; f_{ij}, m_{f_{ij}})$. The other sense is just (i).

Proposition 6.2.4. *In* DEN, $(f_i, g_i) \in I(x_i, D_i; x, D)$ $(i = 1, 2, 3)$, *then* $(f_1, g_1) \,\&\, (f_2, g_2) = (f_3, g_3)$ *iff*

(i) $\mathrm{rg}(f_1) \cap \mathrm{rg}(f_2) = \mathrm{rg}(f_3)$ (*i.e.*, $f_1 \,\&\, f_2 = f_3$);

(ii) $\mathrm{rg}(g_1) \cap \mathrm{rg}(g_2) = \mathrm{rg}(g_3)$.

Proof. (1) Assume that $(f_1, g_1) \,\&\, (f_2, g_2) = (f_3, g_3)$. So we have (f_{31}, g_{31}) and (f_{32}, g_{32}) such that $(f_3, g_3) = (f_1 f_{31}, g_1 g_{31}) = (f_2 f_{32}, g_2 g_{32})$. From this

$$\mathrm{rg}(f_3) \subset \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2), \qquad \mathrm{rg}(g_3) \subset \mathrm{rg}(g_1) \cap \mathrm{rg}(g_2).$$

Define x_3' and $f_3' \in I(x_3', x)$ by $\mathrm{rg}(f_3') = \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2)$, hence $f_3' = f_1 f_{31}' = f_2 f_{32}'$ for appropriate f_{31}' and f_{32}'. Similarily, let $\Delta = \mathrm{rg}(g_1) \cap \mathrm{rg}(g_2)$, and define D_3' by $s \in D_3'^*$ iff $m_{f_3'}(s) \in \Delta$. Define g_3' by $g_3'(s) = m_{f_3'}(s)$, so $(f_3', g_3') \in I(x_3', D_3'; x, D)$, and remark that $g_3' = g_1 g_{31}' = g_2 g_{32}'$ for appropriate g_{31}' and g_{32}'. So Definition 1.5.1(i) is satisfied by $(x_3', D_3'; f_3', g_3'; f_{31}', g_{31}'; f_{32}', g_{32}')$. Condition 1.5.2(ii) yields $(h, k) \in I(x_3', D_3'; x_3, D_3)$ such that $(f_{31}', g_{31}') = (f_{31}h, g_{31}k)$ and $(f_{32}', g_{32}') = (f_{32}h, g_{32}k)$, so $(f_3', g_3') = (f_3 h, g_3 k)$, so $\mathrm{rg}(f_3') \subset \mathrm{rg}(f_3)$, $\mathrm{rg}(g_3') \subset \mathrm{rg}(g_3)$, so (i) and (ii) hold.

(2) Assume that (i) and (ii) hold, then define (f_{31}, g_{31}) and (f_{32}, g_{32}) by $(f_3, g_3) = (f_1 f_{31}, g_1 g_{31}) = (f_2 f_{32}, g_2 g_{32})$. Assume now that $(x_3', D_3'; f_3', g_{31}'; f_{32}', g_{32}')$ is any solution of Definition 1.5.1(i), then

$$\mathrm{rg}(f_3') \subset \mathrm{rg}(f_1) \cap \mathrm{rg}(f_2), \qquad \mathrm{rg}(g_3') \subset \mathrm{rg}(g_1) \cap \mathrm{rg}(g_2).$$

It is therefore possible to define $(h, k) \in I(x_3', D_3'; x_3, D_3)$ by: $(f_3', g_3') = (f_3 h, g_3 k)$. In fact (h, k) is uniquely determined by this condition, which, in turn implies $(f_{31}, g_{31}') = (f_{31}h, g_{31}k)$ and $(f_{32}', g_{32}') = (f_{32}h, g_{32}k)$ (for instance, from $f_3' = f_3 h$, one deduces $f_1 f_{31} = f_1 f_{31} h$, and, since f_1 is injective, $f_{31}' = f_{31} h$).

Corollary 6.2.5. (i) *the functor $\underline{t}$ commutes to pull-backs;*

(ii) *if $(x_i, D_i) = f_i^{-1}(x, D)$ $(i = 1, 2, 3)$, then $(f_1, m_{f_1}) \,\&\, (f_2, m_{f_2}) = (f_3, m_{f_3})$ iff $f_1 \,\&\, f_2 = f_3$.*

Proof. (i) This is obvious from Proposition 6.5.4.

(ii) If $f_1 \,\&\, f_2 = f_3$, then $\mathrm{rg}(f_1) \cap \mathrm{rg}(f_2) = \mathrm{rg}(f_2) = \mathrm{rg}(f_3)$. Furthermore, it is obvious that $\mathrm{rg}(m_{f_1}) \cap \mathrm{rg}(m_{f_2}) = \mathrm{rg}(m_{f_3})$. So by Proposition 6.2.4, $(f_1, m_{f_1}) \,\&\, (f_2, m_{f_2}) = (f_3, m_{f_3})$.

Definition 6.2.6. (i) If (x, D) is a dendroid, then the *height* of (x, D) is by definition the ordinal isomorphic with the well-order of D defined by:

$$t \neq u \rightarrow (s * (t) * s' <^* s * (u) * s'' \leftrightarrow t < u).$$

(ii) The functor *height* from DEN to ON is defined by: $\underline{h}(x, D) = \text{height of } (x, D)$.

$\underline{h}(f, g)(a) = b$ iff the image under g of the ath element of D (for the order $<^*$) is the bth element of D' (if $(f, g) \in I(x, D; x', D')$).

Remark 6.2.7. The fact that $<^*$ is a well-ordering is a triviality. If (s_n) is a strictly decreasing sequence in D, with $s_n = (a_0^n, \ldots, a_{p-1}^n)$, then $p_n \neq 0$ for all n, and (a_0^n) is a decreasing sequence of ordinals. So, for $n \geq N_0$, $a_0^n = A_0$. For $n \geq N_0$, $p_n \neq 1$, and $(a_1^n)_{n \geq N_0}$ is a decreasing sequence of ordinals, so for $n \geq N_1 \geq N_0$, $a_1^n = A_1$. This process gives a sequence (A_p) such that for all p $(A_0, \ldots, A_p) \in D^*$, in contradiction with property (v) of dendroids.

Proposition 6.2.8. *The functor $\underline{h}$ commutes to direct limits and to pull-backs.*

Proof. This is immediate from the characterisations of direct limits and pull-backs by means of the ranges.

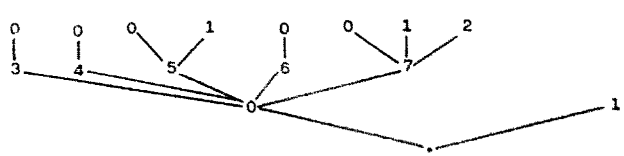
6.3. Homogeneity, strong homogeneity

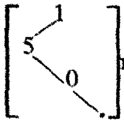
Definition 6.3.1. A dendroid (x, D) is *homogeneous* iff:

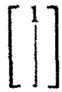
($*$) for all $x' \leq x$, $f, g \in I(x', x)$, $f^{-1}(D) = g^{-1}(D)$;

($**$) for all $x' \leq x$, $f, g \in I(x', x)$, for all $s = (x_0, \ldots, x_{k-1}) \in f^{-1}(D)^*$ such that $f(x_1) = g(x_1)$, $f(x_3) = g(x_3), \ldots,$ then $m_f(s) = m_g(s)$.

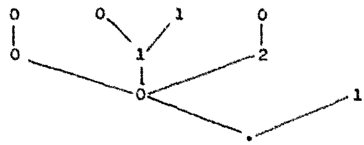
Examples 6.3.2. (i) The following denroid of type 10 is not homogeneous



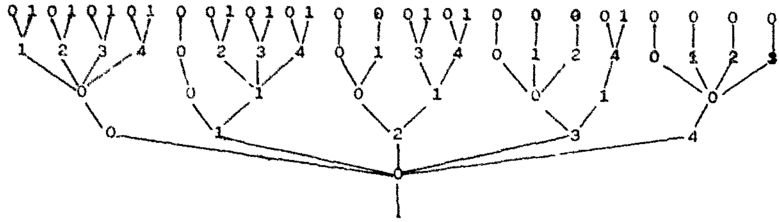
(as usual, we represent finite sequences by branches: the branch  represents the sequence (0, 5, 1)).

If one takes $f \in I(3, 10)$, with $f(0)=1$, $f(1)=2$, $f(2)=9$, then $f^{-1}(D)$ is the dendroid  (this comes from the branch  in D).

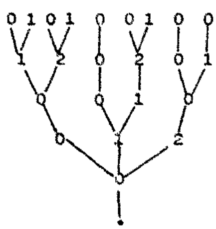
If one takes $f \in I(3, 10)$, with $f(0)=4$, $f(1)=5$, $f(2)=6$, then $f^{-1}(D)$ is



(ii) The following dendroid of type 5 is homogeneous:



For instance, if $f \in I(3, 5)$, then $f^{-1}(D)$ is always equal to:



Proposition 6.3.3. *In Definition* 6.3.1 $(*)$ *and* $(**)$ *can be weakened in* $(*)'$ *and* $(**)'$ *by replacing* '$\forall x' \leq x$' *by* '$\forall x'$ *finite* $\leq x$'.

Proof. Write $(x', h_i) = \varinjlim(n_i, h_{ij})$, and let $f_i = fh_i$, $g_i = gh_i$. Let $D'^* = \mathrm{rg}(m_f)$, $D''^* = \mathrm{rg}(m_g)$, $D_i'^* = \mathrm{rg}(m_{f_i})$, $D_i''^* = \mathrm{rg}(m_{g_i})$. Observe that $D'^* = \bigcup_i D_i'^*$, $D''^* = \bigcup_i D''^*$ ($s = (x_0, \ldots, x_{k-1}) \in D'^*$ iff $x_1, x_3, \ldots \in \mathrm{rg}(f)$, $s \in \bigcup_i D_i'^*$ iff $x_1, x_3, \ldots \in \bigcup_i \mathrm{rg}(f_i) = \mathrm{rg}(f)$). The functions $\varphi_i = m_{g_i} m_{f_i}^{-1}$ (which exist by $(*)'$) are bijections from $D_i'^*$ onto

$D_i''^*$, and are the only such bijections to enjoy Definition 6.1.3(ii)–(v) (with gf_i^{-1} in (iv)). Obviously if $i<j$, then φ_j maps $D_i'^*$ onto $D_i''^*$ and enjoys Definition 6.1.3(ii)–(v). So φ_j extends φ_i. The union φ of all φ_i defines a bijection from D'^* onto D''^* enjoying Definition 6.1.3(ii)–(v) (with gf^{-1} in (iv)), and this is the only such bijection. From this it follows that $f^{-1}(D)=g^{-1}(D)$ and $\varphi=m_g m_f^{-1}$: this proves (∗). If $s=(x_0,\ldots,x_{k-1})\in f^{-1}(D)^*$, and $f(x_1)=g(x_1)$, $f(x_3)=g(x_3),\ldots$, let $t=m_g(s)$. By property (∗∗)′, $\varphi_i(t)=t$ as soon as $t\in D_i'^*$, so $\varphi(t)=t$, but this implies $m_f(s)=m_g(s)$: (∗∗) is proved.

Proposition 6.3.4. *If* (x,D) *is homogeneous, there is a functor* F *from* ON$\leqslant x$ *into* DEN *such that:*

(i) *for all* $x'\leqslant x$, $F(x')$ *is a dendroid of type* x',
(ii) *for all* x', x'', $f\in I(x',x'')$, $F(f)=(f,m_f)$;
(iii) $F(x)=(x,D)$.

Proof. Assuming that (x,D) is homogeneous, then define $F(x')=(E_{x'x})^{-1}(x,D)$. It is immediate that, if $f\in I(x',x'')$ $(f,m_f)\in I(f^{-1}(F(x''));F(x''))$; but

$$f^{-1}(F(x''))=f^{-1}E_{x''x}^{-1}(F(x))=(E_{x''x}f)^{-1}(F(x))=E_{x'x}^{-1}(F(x))=F(x'),$$

by homogeneity. So $(f,m_f)\in I(F(x');F(x''))$ and using the fact that $m_{fg}=m_f m_g$, it is immediate that F is a functor.

Proposition 6.3.5. *Let* (x,D) *be a homogeneous dendroid, and let* F *be the functor associated with* D *by Proposition* 6.3.4, *then* F *commutes to* $\varinjlim$ *and* &.

Proof. F commutes to $\varinjlim$: if $(x,f_i)=\varinjlim(x_i,f_{ij})$, then by Corollary 6.2.3(ii),

$$(F(x);f_i,m_{f_i})=\varinjlim(F(x_i);f_{ij},m_{f_{ij}}),$$

i.e.,

$$(F(x),F(f_i))=\varinjlim(F(x_i),F(f_{ij})).$$

F commutes to &: if $f_1\mathbin{\&}f_2=f_3$, then, by Corollary 6.2.5(ii), $(f_1,m_{f_1})\mathbin{\&}(f_2,m_{f_2})=(f_3,m_{f_3})$, so $F(f_1)\mathbin{\&}F(f_2)=F(f_3)$.

Definition 6.3.6. A dendroid D of type ω is said to be *strongly homogeneous* (in short D is a sh. dendroid) iff for all $x\geqslant\omega$ there is an homogeneous dendroid (x,D') such that $E_{\omega x}^{-1}(D')=D$.

Proposition 6.3.7. D *is strongly homogeneous iff there is a functor* D^0 *from* ON *to* DEN *such that:*

(i) *for all* x, $D^0(x)$ *is an homogeneous dendroid of type* x;
(ii) *for all* x, y, f, $f\in I(x,y)$, $D^0(f)=(f,m_f)$;
(iii) $D^0(\omega)=(\omega,D)$.

Proof. (1) If such a functor exists, then $(E_{\omega x})^{-1}D^0(x) = D^0(\omega) = D$.

(2) Conversely, assume that D is strongly homogeneous, then $D^0(x) = (x, D')$ is uniquely determined by the condition $(E_{\omega x})^{-1}(D') = D$. Since D' is homogeneous, there is a functor F from ON$\leqslant x$ to DEN enjoying Proposition 6.3.4, with D replaced by D'. F commutes to $\varinjlim$, by Proposition 6.3.5, so $F(x) = \varinjlim^*(F(n_i), F(f_{ij}))$, for (n_i, f_{ij}) such that $x = \varinjlim^*(n_i, f_{ij})$, and since the values $F(n_i)$ are uniquely determined by $F(\omega) = D$, as well as $F(f_i) = (f_{ij}, m_{f_{ij}})$, and in DEN we have unicity of direct limits (because there is no non trivial isomorphism) it follows that the value $F(x)$ is uniquely determined.

Define D^0 to be the union of all functors F defined as above, when x varies through ON: properties 6.3.4 of these functors F give (i)–(iii) above.

Remark 6.3.8. We have seen that D^0 is uniquely determined, when it exists by Proposition 6.3.7(i)–(iii). Also, observe that D^0 commutes to $\varinjlim$ and &, by construction and Proposition 6.3.5.

Definition 6.3.9. The following data define a category SHD:

objects: strongly homogeneous dendroids;

morphisms from D to D': the set $I_{\mathrm{sh}}(D, D')$ of functions g from D^* to D'^* such that $(E_\omega, \mathrm{g}) \in I(\omega, D; \omega, D')$, and $m_h^{D'}\mathrm{g} = \mathrm{g}m_h^D$ for all $h \in I(\omega, \omega)$.

Proposition 6.3.10. *Suppose that D and D' are* sh. *dendroids, then*

(i) *if T is a natural transformation from D^0 to D'^0, then for all x, $T(x) = (E_x, T_x)$ for some T_x. So $T_\omega \in I_{\mathrm{sh}}(D, D')$;*

(ii) *conversely, given $\mathrm{g} \in I_{\mathrm{sh}}(D, D')$, there is an unique natural transformation g^0 from D^0 to D'^0, such that $\mathrm{g}^0(\omega) = (E_\omega, \mathrm{g})$.*

Proof. (i) $\underline{t} \circ T$ is a natural transformation from $\underline{t} \circ D^0$ to $\underline{t} \circ D'^0$. But $t \circ D^0 = \underline{t} \circ D'^0 = \mathrm{Id}$, so $\underline{t} \circ T \in I(\mathrm{Id}, \mathrm{Id})$. But there is exactly one natural transformation U in $I(\mathrm{Id}, \mathrm{Id})$, and this is the identity (because $\mathrm{rg}(\mathrm{Id}) = \{(0; 1)\}$, so $\mathrm{rg}(U) = \{(0; 1)\}$, see Section 4.2). So $\underline{t}(T(x)) = E_x$ for all x, i.e., $T(x) = (E_x, T_x)$. $m_h T_\omega = T_\omega m_h$ is obvious.

(ii) Assume that M and M' are the mutilation functions associated with $E_{n\omega}$ (so M and M' are denoted both by $m_{E_{n\omega}}$, with our conventions) from respectively $D^0(n)^*$ and $D'^0(n)^*$ to D^* and $D^{*\prime}$. Let $X \subset D^*$ and $X' \subset D'^*$ be the respective ranges of M and M': $(a_0, \ldots, a_{2p}) \in X$ (resp. $\in X'$) iff it belongs to D (resp. D') and $a_1, a_3, a_5, \ldots < n$. Hence g maps X into X' (Definition 6.1.3(iv)), so it is possible to define a function T_n from $D^0(n)^*$ to $D'^0(n)^*$ by: $M'(T_n(s)) = \mathrm{g}(M(s))$, and it is immediate that $(E_n, T_n) \in I(D^0(n), D'^0(n))$. If $f \in I(n, m)$, then (here we use again the ambiguous notation m_f, $m_{E_{m\omega}}$), with $h \in I(\omega, \omega)$ such that $hE_{n\omega} = E_{m\omega}f$:

$$m_{E_{m\omega}} m_f T_n = m_h m_{E_{n\omega}} T_n = m_h \mathrm{g} m_{E_{n\omega}} = \mathrm{g} m_h m_{E_{n\omega}} = \mathrm{g} m_{E_{m\omega}} m_f = m_{E_{m\omega}} T_m m_f,$$

so $m_f T_n = T_m m_f$ (the equality $m_h g = g m_h$ is needed in the third equality above). So, it follows that $T(n) = (E_n, T_n)$ defines a natural transformation from the functor $D^0 \upharpoonright \mathrm{ON}{<}\omega$ to $D'^0 \upharpoonright \mathrm{ON}{<}\omega$. By Theorem 2.1.5(ii) T can be uniquely extended into a natural transformation g^0 from D^0 to D'^0 (of course Theorem 2.1.5(ii) cannot be applied exactly, since DEN is not closed under direct limits, but the inspection of the proof shows that our deduction is correct).

Remark 6.3.11. The monstrous condition $m_h g = g m_h$ for all $h \in I(\omega, \omega)$ can obviously be replaced by a condition involving only the sets $I(n, \omega)$.

Remark 6.3.12. If one wants to make a category with the homogeneous dendroids, then it is necessary to take as morphisms from (x, D) to (y, D'): the void set if $x \neq y$, and if $x = y$, the set $I_h(x, D; x, D')$ consisting of all functions f such that $(E_x, f) \in I(x, D{:}\ x, D')$ and such that for all $x' \leq x$ (as usual, x' finite suffices) and $h \in I(x', x)$ $f m_h$ depends only on x'. With this definition, $I_{sh}(D, D') = I_h(\omega, D; \omega, D')$.

Theorem 6.3.13. *It is possible to define a functor* LIN (*linearization*) *from* SHD *to* DIL *by:*

(i) *if D is a sh. dendroid, then* $\mathrm{LIN}(D)(x) = \underline{h}(D^0(x))$, *and* $\mathrm{LIN}(D)(f) = \underline{h}(D^0(f))$;

(ii) *if* $g \in I_{sh}(D, D')$, *then* $\mathrm{LIN}(g)(x) = \underline{h}(g^0(x))$.

Proof. By Proposition 6.2.8 and Remark 6.3.8, LIN(D) commutes to $\varinjlim$ and &. So LIN(D) is a dilator. The fact that LIN is a functor is immediate.

6.4. The equivalences $\sim_k$

Proposition 6.4.1. *Let* $(a; n)$, $(b; m)$ *be distinct elements of* $\mathrm{rg}(F)$, *where F is a dilator (see Definition 4.2.2). Define p to be the greatest integer enjoying:*

(P) $\quad \forall i, j < p \quad \sigma(i) < \sigma(j) \leftrightarrow \tau(i) < \tau(j)$

with $\sigma = \sigma_{a,n}$, $\tau = \sigma_{b,m}$ *(see Definition 3.2.8). Assume that* $x_0 < \cdots < x_{n-1} < x$, $y_0 < \cdots < y_{m-1} < x$ *and* $x_{\sigma(0)} = y_{\tau(0)}, \ldots, x_{\sigma(p-1)} = y_{\tau(p-1)}$, *then the order relation between* $(a; x_0, \ldots, x_{n-1}; x)$ *and* $(b; y_0, \ldots, y_{m-1}; x)$ *is independant of the ordinals* $x_{\sigma(p)}, \ldots, x_{\sigma(n-1)}, y_{\tau(p)}, \ldots, y_{\tau(m-1)}$.

Proof. Assume for instance that

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x)$$

and let $x'_0 < \cdots < x'_{n-1} < x$, $y'_0 < \cdots < y'_{m-1} < x$, be such that $x'_{\sigma(0)} = x_{\sigma(0)}, \ldots, x'_{\sigma(p-1)} = x_{\sigma(p-1)}$, $y'_{\tau(0)} = y_{\tau(0)}, \ldots, y'_{\tau(p-1)} = y_{\tau(p-1)}$. We show that

$$(a; x'_0, \ldots, x'_{n-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x).$$

(i) If $n = p$, then the order relation between the points x_i and y_i are the same as the order relations between the points x'_i and y'_i; by Proposition 2.3.17, one gets $(a; x'_0, \ldots, x'_{n-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x)$; the case $m = p$ is similar.

(ii) If $p < m, n$, let $k = \sigma(p)$, $k' = \tau(p)$. Since p has been chosen maximum with property (P), it follows that, for some $i < p$ either

$$x_k < x_{\sigma(i)} = y_{\tau(i)} < y_{k'}, \qquad \text{(Subcase (a))}$$

or

$$y_{k'} < y_{\tau(i)} = x_{\sigma(i)} < x_k \qquad \text{(Subcase (b))}$$

Subcase (a): Obviously $x'_k < x'_{\sigma(i)} = y'_{\tau(i)} < y'_{k'}$;
–if $x'_k < x_k$ and $y_{k'} < y'_{k'}$, then, by Theorem 3.2.4,

$$(a; x'_0, \ldots, x'_{n-1}; x) < (a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x)$$

–in general, construct an ordinal x', functions $f, g \in I(x, x')$ such that $f(x_{\sigma(i)}) = g(x_{\sigma(i)})$ for $i < p$, and $f(x'_k) < g(x_k)$, $g(y_{k'}) < f(y'_{k'})$ (the existence of x', f, g is immediate from $x'_k, x_k < y'_{k'}, y_{k'}$), and we get

$$(a; f(x'_0), \ldots, f(x'_{n-1}); x') < (b; f(y'_0), \ldots, f(y'_{m-1}); x'),$$

and from this, we get

$$(a; x'_0, \ldots, x'_{n-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x).$$

Subcase (b): Symmetric to Subcase (a).

Proposition 6.4.2. *Let* $(a; n)$, $(b; m)$ *be distinct elements of* rg(F), *where* F *is a dilator, and let* p *be an integer enjoying* (P) (*see Proposition* 6.4.1). *Assume that, for some sequences* $x_0 < \cdots < x_{n-1} < x$, $y_0 < \cdots < y_{m-1} < x$, *enjoying* $x_{\sigma(0)} = y_{\tau(0)}, \ldots, x_{\sigma(k-1)} = y_{\tau(k-1)}$, *one has* $(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x)$, *then, if* $k < p$, *and if* $x'_0 < \cdots < x'_{n-1} < x$, $y'_0 < \cdots < y'_{m-1} < x$ *and* $x'_{\sigma(0)} = y'_{\tau(0)}, \ldots, x'_{\sigma(k-1)} = y'_{\tau(k-1)}$, $x'_{\sigma(k)} < y'_{\tau(k)}$, *we have:*

$$(a; x'_0, \ldots, x'_{n-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x).$$

Proof. –If $x_{\sigma(0)} = x'_{\sigma(0)}, \ldots, x_{\sigma(k-1)} = x'_{\sigma(k-1)}$, and $x'_{\sigma(k)} < x_{\sigma(k)}$, $y_{\tau(k)} < y'_{\tau(k)}$, then, by Theorem 3.2.4, we get

$$(a; x'_0, \ldots, x'_{n-1}; x) < (a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x).$$

–In the general case, one constructs easily an ordinal x', together with $f, g \in I(x, x')$, such that $f(x_{\sigma(0)}) = g(x'_{\sigma(0)}), \ldots, f(x_{\sigma(k-1)}) = g(x'_{\sigma(k-1)})$, and $g(x'_{\sigma(k)}) < f(x_{\sigma(k)})$, $f(y_{\tau(k)}) < g(y'_{\tau(k)})$ (the existence of x', f, g is immediate from the hypothesis $x'_{\sigma(k)} < y'_{\tau(k)}$). Then, by the subcase above, one gets:

$$(a; g(x'_0), \ldots, g(x'_{n-1}); x) < (b; g(y'_0), \ldots, g(y'_{m-1}); x'),$$

which implies

$$(a; x'_0, \ldots, x'_{n-1}; x) < (b; y'_0, \ldots, y'_{m-1}; x).$$

Definition 6.4.3. If $(a; n)$ and $(b; m)$ are distinct elements of rg(F), where F is a dilator, we introduce $\S(a, n; b, m)$ to be the pair (p, ε), with:

(i) p is the smallest integer enjoying (P) of Proposition 6.4.1, and such that for all $x_0 < \cdots < x_{n-1} < x$, $y_0 < \cdots < y_{m-1} < x$, with $x_{\tau(i)} = y_{\sigma(i)}$ for all $i < p$, the order relation between $(a; x_0, \ldots, x_{n-1}; x)$ and $(b; y_0, \ldots, y_{m-1}; x)$ is independant of $x_{\sigma(p)}, \ldots, x_{\sigma(n-1)}, y_{\tau(p)}, \ldots y_{\tau(m-1}$ (such a p exists by Proposition 6.4.1).

(ii) $\varepsilon = +$ if, whenever $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}, x$ are as above, then

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x).$$

(iii) $\varepsilon = -$ if, whenever $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}, x$ are as in (i), then

$$(b; y_0, \ldots, y_{m-1}; x) < (a; x_0, \ldots, x_{n-1}; x).$$

Remarks 6.4.4. (i) If $\S(a, n; b, m) = (p, +)$, then $\S(b, m; a, n) = (p, -)$, and conversely.

(ii) It is convenient to put $\S(a, n; a, n) = (\infty, +)$ (the symbol + here plays no role, and property (i) is no longer true).

(iii) In order to avoid terrible circumlocutions, we shall use the abbreviation $p < \S(a, n; b, m)$ to mean that $\S(a, n; b, m) = (q, \varepsilon)$, and that $p < q$. One uses also $p \leqslant \S(a, n; b, m)$.

(iv) The definition of $\S(a, n; b, m)$ is perfectly finitistic. If $\S(a, n; b, m) = (p, \varepsilon)$, then p will be computable as follows: by Proposition 2.3.17 $(a; x_0, \ldots, x_n; x)$ and $(b; y_0, \ldots, y_{m-1}; x)$ are ordered in the same way as $(a; P_0, \ldots, P_{n-1}; n+m)$ and $(b; Q_0, \ldots, Q_{m-1}; n+m)$, for sequences $P_0, \ldots, P_{n-1}, Q_0, \ldots, Q_{m-1}$ enjoying $P_i < Q_j$ iff $x_i < y_j$. So, in order to compute p, it is sufficient to look to the finitely many cases corresponding to the value $x = n+m$; the same process determines ε.

Theorem 6.4.5. *From the following data (relative to a given dilator F):*

(i) $\sigma_{a,n}$ *and* $\sigma_{b,m}$;

(ii) $\S(a, n; b, m)$;

(iii) *the relative order of the points* $x_0, \ldots, x_{n-1}$ *and* $y_0, \ldots, y_{m-1}$.

Then the order relation between $(a; x_0, \ldots, x_{n-1}; x)$ *and* $(b; y_0, \ldots, y_{m-1}; x)$ *is effectively computable. More precisely, we shall write down explicitely an algorithm which allows us to compute this order relation from* (i)–(iii), *independently of F.*

Proof. If $(a; n) = (b; m)$, then the theorem is a consequence of Theorem 3.2.4. So we assume that $(a; n) \neq (b; m)$. Let k be the smallest integer such that $x_{\sigma(k)} \neq y_{\tau(k)}$ (in that case, we assume, by symmetry, that $x_{\tau(k)} < y_{\tau(k)}$) if there is one such k, $k = \inf(n, m)$ otherwise.

(i) If $\S(a, n; b, m) = (p, +)$, with $k \geqslant p$, then, the definition of p yields

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x).$$

(ii) If $\S(a, n; b, m) = (p, -)$ and $k \geqslant p$, then the definition of p yields

$$(b; y_0, \ldots, y_{m-1}; x) < (a; x_0, \ldots, x_{n-1}; x).$$

(iii) If $\S(a, n; b, m) = (p, \varepsilon)$ and $p > k$, then $x_{\sigma(k)} < y_{\tau(k)}$ by hypothesis. The definition of p implies the existence of an ordinal x' and of sequences $x'_0 < \cdots < x'_{n-1} < x'$, $y'_0 < \cdots < y'_{m-1} < x'$, with $x'_{\sigma(0)} = y'_{\tau(0)}, \ldots, x'_{\sigma(k)} = y'_{\tau(k)}$, such that $(a; x'_0, \ldots, x'_{n-1}; x') < (b; y'_0, \ldots, y'_{m-1}; x')$. So, with $x'' = \sup(x, x')$

$$(a; x'_0, \ldots, x'_{n-1}; x'') < (b; y'_0, \ldots, y'_{m-1}; x'').$$

Proposition 6.4.2 yields

$$(a; x_0, \ldots, x_{n-1}; x'') < (b; y_0, \ldots, y_{m-1}; x''),$$

and, from this,

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x).$$

Theorem 6.4.6. *If* $\S(a, n; b, m) = (p, +)$, $\S(b, m; c, l) = (q, +)$, *then* $\S(a, n; c, l) = (\inf(p, q), +)$.

Proof. We prove the theorem in the non trivial case $(a; n) \neq (b; m) \neq (c; l)$. Assume that $\S(a, n; c, l) = (r, \varepsilon)$, then

(i) $r \geqslant \inf(p, q)$: if $s = \inf(p, q)$, one can assume $s \neq 0$, and let $\sigma = \sigma_{a,n}$, $\tau = \sigma_{a,n}$, $\tau = \sigma_{b,m}$, $\rho = \sigma_{c,l}$, and choose sequences $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{m-1}$, $z_0, \ldots, z_{l-1}$, bounded by $x = n + m + l$ with $x_{\sigma(0)} = y_{\tau(0)} = z_{\rho(0)}, \ldots, x_{\sigma(s-2)} = y_{\tau(s-2)} = z_{\rho(s-2)}$ and
–either $x_{\sigma(s-1)} < y_{\tau(s-1)} < z_{\rho(s-1)}$. By Theorem 6.4.5 one gets

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x) < (c; z_0, \ldots, z_{l-1});$$

–or $x_{\sigma(s-1)} > y_{\tau(s-1)} > z_{\rho(s-1)}$. By Theorem 6.4.5, one gets

$$(a; x_0, \ldots, x_{n-1}; x) > (b; y_0, \ldots, y_{m-1}; x) > (c; z_0, \ldots, z_{l-1}; x).$$

This shows that the ordering of $(a; x_0, \ldots, x_{n-1}; x)$ and $(c; z_0, \ldots, z_{l-1}; x)$ depends on the relative order of $x_{\sigma(s-1)}$ and $z_{\rho(s-1)}$. So $r \geqslant s$.

In order to prove that $\inf(p, q) \geqslant r$, assume that $x_0, \ldots, x_{n-1}$, $z_0, \ldots, z_{l-1}$ enjoy $x_{\sigma(0)} = z_{\rho(0)}, \ldots, x_{\sigma(s-1)} = z_{\rho(s-1)}$. We shall prove that $(a; x_0, \ldots, x_{n-1}; x) < (c; z_0, \ldots, z_{l-1}; x)$; this, together with (i) above, will entail $\S(a, n; c, l) = (s, +)$. Remark that, by Proposition 2.3.17, one can suppose that $x_0, \ldots, x_{n-1}$, $z_0, \ldots, z_{l-1}, x$ are limit ordinals (this will make the interpolation by means of $y_0, \ldots, y_{m-1}$ possible). (ii) If $p = q$, choose a sequence $y_0, \ldots, y_{m-1}$, with $x_{\sigma(0)} = y_{\tau(0)}, \ldots, x_{\sigma(p-1)} = y_{\tau(p-1)}$; conclude, with the help of Theorem 6.4.5, that

$$(a; x_0, \ldots, x_{n-1}; x) < (b; y_0, \ldots, y_{m-1}; x) < (c; z_0, \ldots, z_{l-1}; x).$$

(iii) if $p<q$, choose a sequence $y_0, \dots, y_{m-1}$, with $x_{\sigma(0)} = y_{\tau(0)}, \dots, x_{\sigma(p-1)} = y_{\tau(p-1)}$, $y_{\tau(p)} < z_{\rho(p)}$. Again by Theorem 6.4.5:

$$(a; x_0, \dots, x_{n-1}; x) < (b; y_0, \dots, y_{m-1}; x) < (c; z_0, \dots, z_{l-1}; x).$$

(iv) If $q<p$, choose a sequence $y_0, \dots, y_{m-1}$, such that $x_{\sigma(0)} = y_{\tau(0)}, \dots, x_{\sigma(q-1)} = y_{\tau(q-1)}$, $x_{\sigma(q)} < y_{\tau(q)}$. Once again, Theorem 6.4.5 yields

$$(a; x_0, \dots, x_{n-1}; x) < (b; y_0, \dots, y_{m-1}; x) < (c; z_0, \dots, z_{l-1}; x).$$

Remarks 6.4.7. (i) If $\S(a, n; b, m) = (p, \varepsilon)$, if $\S(b, m; c, l) = (q, \varepsilon')$, then $\S(a, n; c, l) = (r, \varepsilon'')$, with $r \geqslant \inf(p, q)$. This comes from part (i) of the proof above.

(ii) So $\underline{d}(a, n; b, m) = 2^{-p}$ (when $\S(a, n; b, m) = (p, \varepsilon)$) defines a metric on $\mathrm{rg}(F)$. This metric enjoys the ultrametric inequality:

$$\underline{d}(a, n; c, l) \leqslant \sup(d(a, n; b, m), d(b, m; c, l)),$$

and is discrete.

Definition 6.4.8. For each integer p, one defines a binary relation $\sim_p^{F,x}$ (often denoted $\sim_p$) on $F(x)$:

$$(a; x_0, \dots, x_{n-1}; x) \sim_p (b; y_0, \dots, y_{m-1}; x)$$

iff:

(i) $\S(a, n; b, m) > p/2$;

(ii) if $\sigma = \sigma_{a,n}$, $\tau = \tau_{b,m}$, then for all $i < p/2$, $x_{\sigma(i)} = y_{\tau(i)}$.

Proposition 6.4.9. *$\sim_p$ is an equivalence relation.*

Proof. Immediate from Remark 6.4.7(i).

Remark 6.4.10. The relations $\sim_p$ generalize the relation $\equiv$ of Theorem 3.1.5. Ones sees easily that $(z, x) \equiv (z', x)$ iff $z \sim_0 z'$.

Theorem 6.4.11. *For all p, the equivalence classes modulo $\sim_p$ are intervals.*

Proof. By induction on p: assume that $A = (a; x_0, \dots, x_{n-1}; x)$ $B = (b; y_0, \dots, y_{m-1}; x)$, $C = (c; z_0, \dots, z_{l-1}; x)$, and that $A \sim_p C$, $A \leqslant B \leqslant C$. We show that $A \sim_p B$. Assume for contradiction that $A \not\sim_p B \not\sim_p C$, then

(i) if $p = 0$, then $\S(a, n; b, m) = (0, \varepsilon)$, $\S(b, m; c, l) = (0, \varepsilon')$. But the definition of $\S$ (Definition 6.4.3) forces $\varepsilon = \varepsilon' = +$, so, by Theorem 6.4.6 one gets $\S(a, n; c, l) = (0, +)$, a contradiction,

(ii) if $p = 2q+1$, the induction hypothesis yields $A \sim_{2q} B \sim_{2q} C$. So the hypothesis $A \not\sim_p B \not\sim_p C$ implies $x_{\sigma(q)} \neq y_{\tau(q)} \neq z_{\rho(q)}$. If $y_{\tau(q)} < x_{\sigma(q)}$, then, since $x_{\sigma(0)} = y_{\tau(0)}, \dots, x_{\sigma(q-1)} = y_{\tau(q-1)}$, and $\S(a, n; b, m) > q$, it follows by Theorem 6.4.5, that $B < A$. So $x_{\sigma(q)} < y_{\tau(q)}$. Similarily $y_{\tau(q)} < z_{\rho(q)}$, so $x_{\sigma(q)} < z_{\rho(q)}$, contradiction with $A \sim_p C$, which implies $x_{\sigma(q)} = z_{\rho(q)}$;

(ii) if $p=2q+2$, the induction hypothesis yields $A \sim_{2q+1} B \sim_{2q+1} C$; so the hypothesis $A \not\sim_p B \not\sim_p C$ implies $\S(a,n;b,m)=(q+1,\varepsilon)$ and $\S(b,m;c,l)=(q+1,\varepsilon')$. Since $x_{\sigma(0)}=y_{\tau(0)},\ldots,x_{\sigma(q)}=y_{\tau(q)}$, it follows that $\varepsilon=+$; similarily, $\varepsilon'=+$, so, by Theorem 6.4.6 $\S(a,n;b,l)=(q+1,+)$, a contradiction with $A\sim_{2q+2} C$.

Proposition 6.4.12. (i) *If $f\in I(x,y)$ and $z\sim_p^{F,x} z'$, then $F(f)(z)\sim_p^{F,y} F(f)(z')$.*

(ii) *If $T\in I(F,G)$ and $z\sim_p^{F,x} z'$, then $T(x)(z)\sim_p^{G,x} T(x)(z')$.*

(iii) *The equivalence class of $(a;x_0,\ldots,x_{n-1};x)$ modulo $\sim_{2n}^{F,x}$ consists of one point.*

Proof. (i) and (ii) are easy exercises for the reader. For (iii), observe that $\S(a,n;b,m)\leqslant \inf(n,m)$, exept if $(a;n)=(b;m)$. So, if

$$(a;x_0,\ldots,x_{n-1};x)\sim_{2n}(b;y_0,\ldots,y_{m-1};x),$$

one gets $(a;n)=(b,m)$, and $x_0=y_0,\ldots,x_{n-1}=y_{m-1}$.

6.5. The functor branching

Notations 6.5.1. We shall use the notation I_k to denote the pair (I_k,k) where I_k is an equivalence class modulo $\sim_k^{F,x}$ (F and x are supposed to be determined by the context. If this is not clear, we say 'I_k in $F(x)$'). I_k is an interval, and observe that there is an unique sequence such that $I_0\supset\cdots\supset I_k$ (for $i\leqslant k$, I_i is the equivalence class modulo $\sim_i$ containing I_k). Define $I_k<J_k$ by: for all $z\in I_k$, $z'\in J_k$, $z<z'$ and $I_k\leqslant J_k$ to mean $I_k<J_k$ or $I_k=J_k$. Define the ordinals $|I_k|$ by:

$|I_0|=$ order type of the set of predecessors of I_0 for $<$.

$|I_{2k+2}|=$ order type of the set of predecessors of I_{2k+2} for $<$, which are included in I_{2k+1}, where $I_{2k+1}\supset I_{2k+2}$.

$|I_{2k+1}|=$ the ordinal b defined by: if $(a;x_0,\ldots,x_{n-1};x)\in I_{2k+1}$, then $x_{\sigma(k)}=b$.

(If $k\geqslant n$, $|I_{2k+1}|$ is not defined, but this does not matter.)

If I_k is an interval in $F(x)$, and $f\in I(x,y)$, then we denote by $\bar F(f)(I_k)$ the interval in $F(y)$ containing the image of I_k under $F(f)$ and which is an equivalence class modulo $\sim_k^{F,y}$. Similarily, if $T\in I(F,G)$, we denote by $\bar T(x)(I_k)$ the equivalence class modulo $\sim_k^{G,x}$ containing the image of I_k under $T(x)$.

Definition 6.5.2. (i) If F is a dilator, one defines the function $\varphi_{F,x}$ from $F(x)$ into the class of finite sequences of ordinals by:

$$\varphi_{F,x}((a;x_0,\ldots,x_{n-1};x))=(|I_0|,\ldots,|I_{2n}|),$$

where

$$I_{2n}=\{(a;x_0,\ldots,x_{n-1};x)\}\quad\text{and}\quad I_0\supset\cdots\supset I_{2n}.$$

(ii) If F is a dilator, one defines the function $\psi_{F,f}$ from $\mathrm{rg}(\varphi_{F,x})^*$ to $\mathrm{rg}(\varphi_{F,y})^*$, when $f\in I(x,y)$, by:

$$\psi_{F,f}(|I_0|,\ldots,|I_{p-1}|)=(|\bar F(f)(I_0)|,\ldots,|\bar F(f)(I_{p-1})|).$$

(iii) If $T \in I(F, G)$, then one defines the function $\psi_{T,x}$ from $\mathrm{rg}(\varphi_{F,x})^*$ to $\mathrm{rg}(\varphi_{G,x})^*$ by:

$$\psi_{T,x}(|I_0|, \ldots, |I_{p-1}|) = (|\bar{T}(x)(I_0)|, \ldots, |\bar{T}(x)(I_{p-1})|).$$

Theorem 6.5.3. (i) $\mathrm{rg}(\varphi_{F,x})$ *is a dendroid of type* x.

(ii) $(f, \psi_{F,f}) \in I(x, \mathrm{rg}(\varphi_{F,x});\ y,\ \mathrm{rg}(\varphi_{F,y}))$ *and for all* $z \in F(x)$, $\psi_{F,f}\varphi_{F,x}(z) = \varphi_{F,y}F(f)(z)$.

(iii) $(E_x, \psi_{T,x}) \in I(x, \mathrm{rg}(\varphi_{F,x}); x, \mathrm{rg}(\varphi_{G,x}))$, *and for all* $z \in F(x)$, $\varphi_{G,x}T(x)(z) = \psi_{T,x}\varphi_{F,x}(z)$.

Proof. First of all, let us note that the sequence $(|I_0|, \ldots, |I_p|)$ determines completely I_p (this justifies Definition 6.5.2). By induction on p: $|I_0|$ determines I_0, since I_0 is the $|I_0|$th equivalence class. If $p = 2q+1$, then, by induction hypothesis, I_{2q} is known. The ordinal $|I_{2q+1}|$ gives the common values $x_{\sigma(q)}$: $(a; x_0, \ldots, x_{n-1}; x) \in I_{2q+1}$ iff if belongs to I_{2q} and $x_{\sigma(q)} = |I_{2q+1}|$. Finally, if $p = 2q+2$ and by induction hypothesis I_{2q+1} is known, then I_{2q+2} is the $|I_{2q+2}|$th equivalence class modulo $\sim_{2q+2}$ included in I_{2q+1}.

(i) We have to check properties (i)–(v) of dendroids. (i)–(iv) are immediate. For instance (ii) if $z \neq z'$, then the sequences $(I_0, \ldots, I_{2n})$ and $(I'_0, \ldots, I'_{2m})$ differ, since $I_{2n} = \{z\} \neq \{z'\} = I'_{2m}$, so, by the remark above, the sequences $(|I_0|, \ldots, |I_{2n}|)$ and $(|I'_0|, \ldots, |I'_{2m}|)$ are such that $|I_i| \neq |I'_i|$ for some i.

Property (v) is more delicate. A strictly decreasing sequence in $\mathrm{rg}(\varphi_{F,x})^*$, for the relation 'is an extension' is the same thing as a decreasing sequence $I_0 \supset \cdots \supset I_n \supset \cdots$. Define $f \in I(x, \omega x)$ by $f(z) = \omega \cdot z$, and let $J_n = \bar{F}(f)(I_n)$. So $J_0 \supset \cdots \supset J_n \supset \cdots$, and $J_n = [a_n, b_n[$ for all n. The ordinals b_n are decreasing, so $b_n = b = \text{constant}$ for all $n \geq N$. But, if $(z; p_0, \ldots, p_{n-1}; \omega x) = c \in J_{2N+1}$, define $q_0, \ldots, q_{r-1}$:

$$q_{\sigma(0)} = p_{\sigma(0)}, \ldots, q_{\sigma(N-1)} = p_{\sigma(N-1)},$$
$$q_{\sigma(N)} = p_{\sigma(N)} + 1, \ldots, q_{\sigma(n-1)} = p_{\sigma(n-1)} + 1$$

(this is possible, since $p_{\sigma(0)} = |J_1|, \ldots, p_{\sigma(N-1)} = |J_{2N-1}|$ are in $\mathrm{rg}(f)$). If $d = (z; q_0, \ldots, q_{n-1}; \omega x)$, then $c \sim_{2N} d$, so $d < b_{2N}$, but $c \not\sim_{2N+1} d$ and $c < d$, so $d \geq b_{2N+1}$, contradiction with $b_{2N} = b_{2N+1} = b$.

(ii) $\psi_{F,f}\varphi_{F,x}(z) = \varphi_{F,y}F(f)(z)$ is just the definition, so all we need to prove is that $(f, \psi_{F,f})$ is a morphism of dendroids.

(i): Trivial.

(ii): Immediate. If $I_{2n} = \{(z; x_0, \ldots, x_{n-1}; x)\}$, then

$$\bar{F}(f)(I_{2n}) = \{(z; f(x_0), \ldots, f(x_{n-1}); y)\}$$

by Proposition 6.4.12(iii). This shows that $\psi_{F,f}$ maps $\mathrm{rg}(\varphi_{F,x})$ into $\mathrm{rg}(\varphi_{F,y})$.

(iii): Trivial.

(iv): Immediate from the fact that $|\bar{F}(f)(I_{2k+1})| = f(|I_{2k+1}|)$.

(v): Immediate from the fact that $I_{2k} < I'_{2k} \to \bar{F}(f)(I_{2k}) < \bar{F}(f)(I'_{2k})$.
(iii) This case is parallel to (ii).

Theorem 6.5.4. *There exists a functor* BCH *(branching) from* DIL *to* SHD, *defined by:*

$$\mathrm{BCH}(F) = \mathrm{rg}(\varphi_{F,\omega}), \qquad \mathrm{BCH}(T) = \psi_{F,\omega}.$$

Moreover, one has:
(i) $\mathrm{BCH}(F)^0(x) = (x, \mathrm{rg}(\varphi_{F,x}))$;
(ii) $\mathrm{BCH}(F)^0(f) = (f, \psi_{F,f})\ (= (f, m_f))$;
(iii) $\mathrm{BCH}(T)^0(x) = (E_x, \psi_{T,x})$.

Proof. The fact that BCH is a functor is left to the reader. By Proposition 6.3.7 and Definition 6.3.9, in order to prove that BCH(F) is a sh. dendroid, and $\mathrm{BCH}(T) \in I_{\mathrm{sh}}(\mathrm{BCH}(F), \mathrm{BCH}(G))$, it suffices to prove that (i) and (ii) define a functor $\mathrm{BCH}(F)^0$ from ON into DEN, enjoying Proposition 6.3.7(i)-(iii), and that (iii) defines a natural transformation from $\mathrm{BCH}(F)^0$ to $\mathrm{BCH}(G)^0$.

(1) We claim that $(|I_0|, \ldots, |I_{2n}|) \in \mathrm{rg}(\psi_{F,f}) \cap \mathrm{rg}(\varphi_{F,y})$ iff $(|I_0|, \ldots, |I_{2n}|) \in \mathrm{rg}(\varphi_{F,y})$ and $|I_1|, |I_3|, |I_5|, \ldots \in \mathrm{rg}(f)$. Assume that $I_{2n} = \{(z; x_0, \ldots, x_{n-1}; y)\}$, then

$$(|I_0|, \ldots, |I_{2n}|) \in \mathrm{rg}(\psi_{F,f}), \quad \text{iff} \quad I_{2n} = \bar{F}(f)(J_{2n})$$

for some J_{2n} $(= \{(z; f^{-1}(x_0), \ldots, f^{-1}(x_{n-1}); x)\})$. So this is equivalent to $x_0, \ldots, x_{n-1} \in \mathrm{rg}(f)$, which can be written $|I_1|, \ldots, |I_{2n-1}| \in \mathrm{rg}(f)$. This shows clearly that $\mathrm{rg}(\psi_{F,f})$ is equal to the range of the multilation function m_f, so $\psi_{F,f} = m_f$. This proves Proposition 6.3.7(ii).

$\mathrm{rg}(\varphi_{F,x})$ is homogeneous: (*) is trivial, (**) immediate.

(2) Using Definition 6.4.3(ii) and (iii), one gets:

$$\begin{aligned}\psi_{G,f}\psi_{T,x}\varphi_{F,x}(z) &= \psi_{G,f}\varphi_{G,x}T(x)(z) = \varphi_{G,y}G(f)T(x)(z)\\ &= \varphi_{G,y}T(y)F(f)(z) = \psi_{T,y}\varphi_{F,y}F(f)(z) = \psi_{T,y}\psi_{F,f}\varphi_{F,x}(z).\end{aligned}$$

This proves that $\psi_{G,f}\psi_{T,x}(s) = \psi_{T,y}\psi_{F,f}(s)$ for all $s \in \mathrm{BCH}(F)^0(x)$. From this $\psi_{G,f}\psi_{T,x}(s) = \psi_{T,y}\psi_{F,f}(s)$ for all $s \in \mathrm{BCH}(F)^0(x)^*$. This proves that $\mathrm{BCH}(T)^0$ is a natural transformation.

Theorem 6.5.5. (i) $\mathrm{LIN} \circ \mathrm{BCH} = \mathrm{Id}_{\mathrm{DIL}}$.
(ii) $\mathrm{BCH} \circ \mathrm{LIN} = \mathrm{Id}_{\mathrm{SHD}}$.

Proof. (i) The function $\varphi_{F,x}$ is strictly increasing. If $z < z'$, if

$$\varphi_{F,x}(z) = (|I_0|, \ldots, |I_{2n}|), \qquad \varphi_{F,x}(z') = (|I'_0|, \ldots, |I'_{2m}|),$$

choose i minimum such that $I_i \neq I'_i$, then $I_i < I'_i$ (since $z \in I_i$ and $z' \in I'_i$), so $|I_0| = |I'_0|, \ldots, |I_{i-1}| = |I'_{i-1}|$, $|I_i| < |I'_i|$, so $\varphi_{F,x}(z) <^* \varphi_{F,x}(z')$. From this, the order type of $\mathrm{BCH}(F)^0(x) = \mathrm{rg}(\varphi_{F,x})$ is equal to $F(x)$. So $\mathrm{LIN}(\mathrm{BCH}(F))(x) = \underline{h}(\mathrm{BCH}(F)^0(x)) = F(x)$. If $f \in I(x, y)$, we know that for all $z \in F(x)\psi_{F,f}\varphi_{F,x}(z) =$

$\varphi_{F,y}F(f)(z)$. This shows that $\underline{h}(f, \psi_{F,f}) = F(f)$, so

$$\mathrm{LIN}(\mathrm{BCH}(F))(f) = \underline{h}(\mathrm{BCH}(F)^0(f)) = \underline{h}(f, \psi_{F,f}) = F(f).$$

If $T \in I(F, G)$, then, for all $z \in F(x)$, $\psi_{T,x}\varphi_{F,x}(z) = \varphi_{G,x}T(x)(z)$, and this means that $\underline{h}(E_x, \psi_{T,x}) = T(x)$, and from this we conclude that

$$\mathrm{LIN}(\mathrm{BCH}(T))(x) = \underline{h}(\mathrm{BCH}(T)^0(x)) = \underline{h}(E_x, \psi_{T,x}) = T(x).$$

(ii) If D is a sh. dendroid, then we have to study first the dilator $\mathrm{LIN}(D) = F$.

(1) If $s \in D$, then let z be the order type of the set of predecessors of s; w.r.t. the dilator $\mathrm{LIN}(D)$, one can write $z = (a; x_0, \ldots, x_{n-1}; \omega)$; we have also $s = (u_0, \ldots, u_{2m})$. From the equality: $F(f) = \underline{h}(D^0(f))$, it follows that $z \in \mathrm{rg}(F(f))$ iff $s \in \mathrm{rg}(m_f)$, when $f \in I(x, \omega)$. From this it follows that $m = n$, and that $x_0, \ldots, x_{n-1}$ are equal to $u_1, u_3, u_5, \ldots$ listed in increasing order. Define a permutation σ of n, by $\sigma(i) = j$ iff $u_{2i+1} = x_j$. We claim that $\sigma = \sigma_{a,n}$. We have to compare the points $z_f = F(f)(z)$, when $f \in I(\omega, \omega)$. Obviously $z_f < z_g$ iff $s_f <^* s_g$, with $s_f = m_f(s)$. By property (**) of homogeneous dendroids, assume that $f(u_1) = g(u_1), \ldots, f(u_{2k-1}) = g(u_{2k-1})$, then, the values of the sequences s_f and s_g on $0, \ldots, 2k$ coincide. Furthermore, if $f(u_{2k+1}) < g(u_{2k+1})$, it follows that $s_f <^* s_g$. So $z_f < z_g$ iff $f(u_1) = g(u_1), \ldots, f(u_{2k-1}) = g(u_{2k-1})$, $f(u_{2k+1}) < g(u_{2k+1})$ for some $k < n$, equivalently $f(x_{\sigma(0)}) = g(x_{\sigma(0)}), \ldots, f(x_{\sigma(k-1)}) = g(x_{\sigma(k-1)})$, $f(x_{\sigma(k)}) < g(x_{\sigma(k)})$. By Theorem 3.2.4, this proves our claim.

(2) We shall now compute the relations $\sim_k$ in $\mathrm{LIN}(D)(\omega)$. If $s' = (v_0, \ldots, v_{2m}) \in D$, if $z' = (b; y_0, \ldots, y_{m-1}; \omega)$ is the order type of the set of predecessors of s', then we calim that

$$z \sim_k z' \quad \text{iff } u_0 = v_0, \ldots, u_k = v_k.$$

Assume that $z \sim_k z'$. So $x_{\sigma(i)} = y_{\tau(i)}$ for all $i < k/2$, hence $u_{2i+1} = v_{2i+1}$ for all i such that $2i+1 \leqslant k$. Let $r = [k/2]$ (the greatest integer $\leqslant k/2$): by hypothesis, $\S(a, n, b, m) > k/2 \geqslant r$, so $\sigma(i) < \sigma(r)$ iff $\tau(i) < \tau(r)$, for all $i < r$ (with $\sigma = \sigma_{a,n}$, $\tau = \sigma_{b,m}$). So there are functions $f, g, h \in I(\omega, \omega)$ such that $f(x_{\sigma(i)}) = g(x_{\sigma(i)}) = h(x_{\sigma(i)})$ for all $i < r$, and $f(x_{\sigma(r)}) < g(y_{\tau(r)}) < h(x_{\sigma(r)})$. Now, with $z'_f = F(f)(z')$, Corollary 6.2.5 yields: $z_f < z'_g < z_h$. With $s'_f = m_f(s')$, this gives $s_f <^* s'_g <^* s_h$. Property (**) of homogeneous dendroids (since $f(u_{2i+1}) = g(u_{2i+1}) = h(u_{2i+1})$ for all i such that $2i+1 < 2r$) yields:

$$m_f((u_0, \ldots, u_{2r})) = m_g((u_0, \ldots, u_{2r})) = m_h((u_0, \ldots, u_{2r})) \ (=A),$$
$$m_f((v_0, \ldots, v_{2r})) = m_g((v_0, \ldots, v_{2r})) = m_h((v_0, \ldots, v_{2r})) \ (=B).$$

So the inequalities $s_f <^* s'_g <^* s_h$ can be rewritten $A * t_f <^* B * t'_g <^* A * t_h$, and so $A = B$. This forces $u_i = v_i$ for all $i \leqslant 2r$. This proves that $u_0 = v_0, \ldots, v_k$ when k is even. When k is odd, the equality $u_{2r+1} = v_{2r+1}$ has been established too.

Conversely, assume that $u_0 = v_0, \ldots, u_k = v_k$. Observe that $u_{2i+1} < u_{2r+1}$ iff $v_{2i+1} < v_{2r+1}$, for $i < r$. If k is odd, then $k = 2r+1$, and this is clear. If k is even, then by property (iii) of dendroids, the point $u_{2i+1} = v_{2i+1}$ cannot lay inside the interval $[u_{2r+1}, v_{2r+1}]$ or $[v_{2r+1}, u_{2r+1}]$.

So, it is possible to define f, g, h exactly as above, and it is immediate that $s_f <^* s'_g <^* s_h$. So $z_f < z'_g = z_h$. The definition of $\S(a, n; b, m)$ shows that $\S(a, n; b, m) > r$, so $\S(a, n; b, m) > k/2$. Also, the hypothesis can be rewritten $x_{\sigma(i)} = y_{\tau(i)}$ for all $i < k/2$, so $z \sim_k z'$.

(3) Let us now compute the ordinals $|I_k|$. Assume that $z \in I_k$, and let s be as above. We claim that $|I_k| = u_k$:

–if k is odd, then $|I_k| = x_{\sigma(k-1/2)} = u_k$,

–if $k = 0$, then $z \sim_0 z'$ iff $u_0 = v_0$, I_0 is the u_0th class, so $|I_0| = u_0$,

–if $k = 2p+2$, then $z \sim_{2p+1} z'$ iff $u_0 = v_0, \ldots, u_{2p+1} = v_{2p+1}$, so u_k distinguishes between various classes modulo $\sim_k$ included in I_{2p+1}. From this, again $|I_k| = u_k$.

(4) From (3), it is possible to compute BCH(LIN(D)). This dendroid is the range of the function defined by:

$$\lambda(s) = (|I_0|, \ldots, |I_{2n}|), \quad \text{with } I_{2n} = \{z\},$$

so $\lambda(s) = s$ for all $s \in D$. From this BCH(LIN(D)) $= D$.

(5) Finally we compute BCH(LIN(g)) when $g \in I_{sh}(D, D')$. if $g(s) = s''$, and if z'' is the order type of s'' in D', then

$$\mathrm{BCH}(\mathrm{LIN}(g))(s) = (|\bar{T}(\omega)(I_0)|, \ldots, |\bar{T}(\omega)(I_{2n})|), \quad \text{with } T = \mathrm{LIN}(g).$$

But by definition $T(\omega)(z) = z''$, and we know by (3) that (if $z'' \in J_k$ in D') that $|J_k| = w_k$ (if $s'' = (w_0, \ldots, w_{2n})$), so

$$\mathrm{BCH}(\mathrm{LIN}(g))(s) = s'' = g(s).$$

Corollary 6.5.6. *The functors* BCH *and* LIN *commute to direct limits and pull-backs.*

Proof. Trivial from the fact that they are isomorphisms.

Remark 6.5.7. The functor length of Theorem 5.1.1 is related to BCH as follows:

$$\mathrm{LH}(F) = \{x; (x) \in \mathrm{BCH}(F)^*\}, \qquad \mathrm{LH}(T)(x) = y \quad \text{iff } \mathrm{BCH}(T)((x)) = (y).$$

This is immediate from Remark 6.4.10.

7. Quasi and multi-dendroids

7.1. Quasi-dendroids

Definition 7.1.1. A *quasi-dendroid D of type x* is a pair (x, D) where x is an ordinal and:

(i) D is a set of finite sequences $s = (x_0, \ldots, x_n)$, such that, for all $i \leq n$

–either x_i is an ordinal,

–or x_i is a pair $(a_i, -)$, with $a_i < x$. Such a pair will be denoted by $\underline{a_i}$. (For instance $s = (0, \omega, \underline{5}, \omega+1, \underline{\omega+\omega})$).

(ii) If $s = (x_0, \ldots, x_n) \in D$, then, for $m < n$, $(x_0, \ldots, x_m) \notin D$.

(iii) If $D^* = \{(\ )\} \cap \{s; s * s' \in D$ for some $s'\}$, then, if x, x' are ordinals and $s*(x) \in D^*$, then $s*(\underline{x'}) \notin D^*$.

(iv) There is no sequence (x_n) such that for all n, $(x_0, \ldots, x_{n-1}) \in D^*$.

Remarks 7.1.2. (i) We shall abbreviate 'quasi-dendroids' by the letters qd. We shall usually speak of 'the qd. D'.

(ii) Dendroids can be considered as particular qd. If D is a dendroid, define a qd. D' by:

$$D' = \{x_0, \underline{x_1}, x_2, \ldots, \underline{x_{2n-1}}, x_{2n}); (x_0, x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \in D\}.$$

We shall identify dendroids with their associated qd.'s.

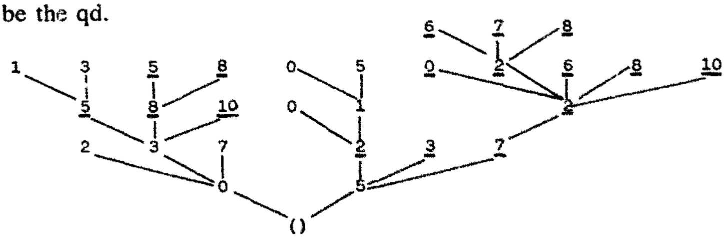
(iii) The *height* $\underline{h}(D)$ of a qd. D is its order type for the order $<^*$ defined as in Remark 6.2.7(i) (when defining $<^*$, we forget the underlinings). There is a bijective function φ_D from D to $\underline{h}(D)$, such that $s <^* s' \leftrightarrow \varphi_D(s) < \varphi_D(s')$.

Remarks 7.1.3. Quasi-dendroids are introduced in order to permit a more flexible use of dendroids. More precisely, we shall define an equivalence relation $\asymp$ between qd., such that

(i) all operations defined on qd. $(\sum, \sum^*, \int \ldots)$ will be compatible with the equivalence,

(ii) each equivalence class modulo $\asymp$ contains one and only one dendroid: this will be done by means of a 'normalization' procedure, which transforms a qd. into a dendroid. Let us start with an example.

Let D be the qd.



and we shall try to construct a (*n* equivalent) dendroid $D' = N(D)$.

(1) First we compute the height of D: $\underline{h}(D) = 16$. D' will have the same height: $\underline{h}(D') = 16$. More precisely, to each $s \in D$, we shall associate a sequence $\nu(s)$, and D' will be $\mathrm{rg}(\nu)$.

(2) When constructing $\nu(s)$ from s, we keep the underlined elements in their order; but, in a dendroid, repetitions of underlined (i.e., of odd index) elements are forbidden, so it is necessary to consider the underlined elements according to their order of apparition, and without repetition.

Definition 7.1.4. (i) If $s=(a_0,\ldots,a_n)$ is a sequence consisting of ordinals and of underlined ordinals, one defines $\mathrm{Occ}(s)$ by: $\mathrm{Occ}(s)=(x_0,\ldots,x_{p-1})$, with

– x_i are pairwise distinct ordinals;

– there is a function $f\in I(p,n+1)$ such that for all $j<n+1$ such that a_j is underlined, there is an $i<p$ such that $a_j=\underline{x}_i=a_{f(i)}$, and $f(i)\leqslant j$.

(ii) We denote by Occ^D the function defined on $\underline{h}(D)$ by:

$$\mathrm{Occ}^D(z)=\mathrm{Occ}(\varphi_D^{-1}(z))$$

(Remark 7.1.3 continued). Hence, for $s\in D$, such that $\mathrm{Occ}(s)=(x_0\ \ldots,x_{p-1})$, $\nu(s)=(b_0,\ldots,b_{2p})$, with $b_{2i+1}=\underline{x}_i$. For instance, let $s=(5,\underline{7},\underline{2},\underline{2},\underline{7})$, then $\mathrm{Occ}(s)=(7,2)$ and

$$\nu(s)=(b_0,\underline{7},b_2,\underline{2},b_4).$$

(3) So, it remains to compute the coefficients b_{2i}, and this will be done by means of equivalence classes $\sim^D_p$. We first make a practical computation:

(a) In D^*, take all maximal sequences without underlinings. We obtain the following sequences in our example:

$$s_0=(0,2),\qquad s_1=(0,3),\qquad s_2=(0,7),\qquad s_3=(5).$$

Then consider in D the classes $C_0,\ldots,C_3$ defined by $C_i=\{s\in D;\ s \text{ extends } s_i\}$. These classes are intervals for $<^*$ (they are the equivalence classes for $\sim^D_0$). This enables us to compute the first coefficient of $\nu(s)$: $\nu(s)=(i)*u(s)$, where i is such that $s\in C_i$. For instance, we obtain:

$$\nu(s)=(3,\underline{7},b_2,\underline{2},b_4),\quad \text{when } s=(5,\underline{7},\underline{2},\underline{2},\underline{7}).$$

(b) Consider now the sequence $t=(5,\underline{7},\underline{2})$. All extensions s of D satisfy $\nu(s)=(3,\underline{7},0,\underline{2})*u(s)$ (we admit that $b_2=0$): we try to compute the first coefficient b_4 of $u(s)$. All coefficients a such that $(5,\underline{7},\underline{2},a)\in D^*$ are underlined, but remark that condition 6.1.1(iii) is not fulfilled. This motivates the following decomposition of $D_t=\{s\in D \text{ and } s \text{ extends } t\}$. Remark that $\mathrm{Occ}(t)=(7,2)$.

We shall classify $s=(5,\underline{7},\underline{2},\underline{a})*s'$ according to the value of a:

– $a<2$: this gives the sequence $s_0=(5,\underline{7},\underline{2},\underline{0})$;

– $a=\underline{2}$: then we have to look to the next underlined point in s, because $\underline{2}$ is not a new underlined point. This gives three classes made of one point:

$$s_1=(5,\underline{7},\underline{2},\underline{2},\underline{6}),\qquad {}_2=(5,\underline{7},\underline{2},\underline{2},\underline{7}),\qquad s_3=(5,\underline{2},\underline{2},\underline{8})$$

(s_1, s_2, s_3 are not equivalent modulo $\sim_4$, because the last coefficient of these three points is in a different situation, w.r.t. 7);

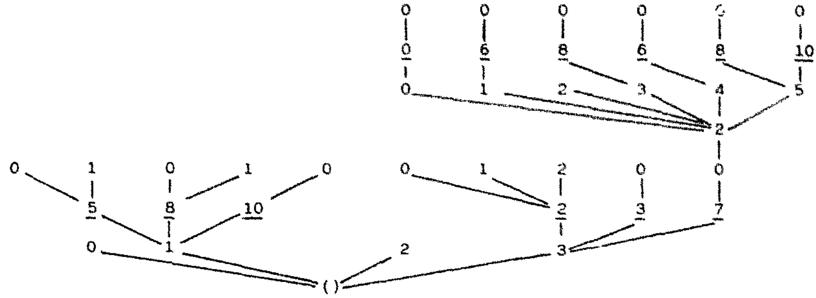
– $2<a<7$: this introduces a class consisting of the only point $s_4=(5,\underline{7},\underline{2},\underline{6})$;

– $a=7$: no point;

– $a>7$ gives a two-point class, consisting of $s_5=(5,\underline{7},\underline{2},\underline{8})$ and $s_6=(5,\underline{7},\underline{2},10)$.

Finally we have obtained six classes modulo $\sim_4$, and these classes yield values

0, 1, 2, 3, 4, 5 for b_4. Finally one obtains D':



It remains to give the precise definition of the relations $\sim_k^D$.

Definition 7.1.5. Let D be qd. and let p be an integer, then

(i) if $s \in D$, then $s \sim_p s$;

(ii) if $s \neq s'$, then: $s \sim_{2p+1} s'$ iff for some t, $s = t * u$, $s' = t * u'$, and $\mathrm{Occ}(t)$ has $p+1$ elements (i.e., $\mathrm{Occ}(t) = (a_0, \ldots, a_p)$) and $s \sim_{2p} s'$ iff for some t, x, x', $s = t*(\underline{x})*u$, $s' = t*(\underline{x}')*u'$, $\mathrm{Occ}(t)$ has p elements (i.e., $\mathrm{Occ}(t) = (a_0, \ldots, a_{p-1})$), and for all $z \in \mathrm{Occ}(t) z \notin [x, x']$.

The equivalences $\sim_k^D$ are also defined on $\underline{h}(D)$ by means of

$$s \sim_k s' \quad \text{iff } \varphi_D(s) \sim_k \varphi_D(s').$$

Definition 7.1.6. We define the following equivalence relation between qd. of the same type: $D \approx D'$ iff:

(1) $\underline{h}(D) = \underline{h}(D')$.

(2) The equivalence relations $\sim_k^D$ and $\sim_k^{D'}$ defined on $\underline{h}(D)$ are the same.

(3) The functions Occ^D and $\mathrm{Occ}^{D'}$, defined on $\underline{h}(D)$ are equal.

Proposition 7.1.7. *On $\underline{h}(D)$, the equivalence classes modulo $\sim_k$ are intervals.*

Proof. (i) If k is odd, let $s \in D$. Then either

$$\mathrm{card}(\{a;\ \underline{a} \text{ occurs in } s\}) < (k+1)/2:$$

the equivalence class of s consists of s alone, or

$$\mathrm{card}(\{a;\ \underline{a} \text{ occurs in } s\}) \geq (k+1)/2:$$

if $s = (x_0, \ldots, x_n)$ choose i minimum such that:

$$\mathrm{card}(\{a;\ a \text{ occurs in } (x_0, \ldots, x_i)\}) = (k+1)/2,$$

then $s \sim_k s'$ iff $s' = (x_0, \ldots, x_i) * t'$. So the equivalence classes modulo $\sim_k$ are of the form $\{t_0 * s;\ t_0 * s \in D\}$. This induces an interval of $\underline{h}(D)$.

(ii) If k is even, we proceed as in (i). If

$$\operatorname{card}(\{a;\ \underline{a} \text{ occurs in } s\}) \leqslant k/2,$$

then the equivalence class of s consists of one element, otherwise, choose i minimum such that

$$\operatorname{card}(\{a;\ \underline{a} \text{ occurs in } (x_0, \ldots, x_i)\}) = k/2 + 1.$$

Let $t = (x_0, \ldots, x_{i-1})$, then $x_i = \underline{a}$ for some a. Let I be the greatest interval containing a, and such that, if $c \in I$, then $\underline{c}$ does not occur in t, then the equivalence class of s is the set

$$\{t * (\underline{c}) * s';\ t * (\underline{c}) * c' \in D \text{ and } c \in I\}.$$

Again the image of this set under φ_D, is an interval.

Remark 7.1.8. If D is a dendroid, then $(x_0, \ldots, x_n) \sim_k^D (y_0, \ldots, y_m)$ iff either $k \leqslant n, m$ and $(x_0, \ldots, x_k) = (y_0, \ldots, y_k)$ or $(x_0, \ldots, x_n) = (y_0, \ldots, y_m)$. Obviously $\operatorname{Occ}(x_0, \ldots, x_{2n}) = (x_1, x_3, \ldots, x_{2n-1})$.

Definition 7.1.9. We use the notation I_k for an equivalence class modulo $\sim_k$ in $\underline{h}(D)$. Then we define the norms $|I_k|$:

(i) $|I_0| =$ order type of $\{I'_0 I'_0 < I_0\}$ ($<$ is defined in Notations 6.5.1).

(ii) $|I_{2k+2}| =$ order type of

$$\{I'_{2k+2};\ I'_{2k+2} < I_{2k+2} \text{ and } I'_{2k+2} \subset I_{2k+1}\},$$

where I_{2k+1} is defined by $I_{2k+2} \subset I_{2k+1}$.

(iii) $|I_{2k+1}| = a_k$, if, for some $x \in I_{2k+1}$, $\operatorname{Occ}(x) = (a_0, \ldots, a_{n-1})$, with $n > k$; $|I_{2k+1}|$ is undefined otherwise.

Remark 7.1.10. The intervals I_k associated with a dendroid are in bijection with all sequences $s = (a_0, \ldots, a_k)$ of D^*:

$$\varphi_D(t) \in I_k \quad \text{iff } \exists s'\ (t = s * s').$$

It is immediate that $|I_k| = x_k$.

Lemma 7.1.11. *Assume that $I_k < I'_k$, and that, if $k \neq 0$, I_k, $I'_k \subset I_{k-1}$, then (provided I_k and I'_k are defined), $|I_k| < |I'_k|$.*

Proof. (i) If k is even, this is simply trivial.

(ii) If $k = 2l + 1$, choose s and s' such that $\varphi_D(s) \in I_k$, $\varphi_D(s') \in I'_k$. Assume that $s = (x_0, \ldots, x_n)$, $s' = (x'_0, \ldots, x'_m)$, $\operatorname{Occ}(s) = (a_0, \ldots, a_{p-1})$, $\operatorname{Occ}(s') = (a'_0, \ldots, a'_{q-1})$. Since I_k and I'_k are defined, it follows that $p, q > l$, choose i and j minimum such that $x_i = \underline{a}_l$, $x'_j = \underline{a}'_l$, then, $s \sim_{k-1} s'$ forces $i = j$ and $x_0 = x'_0, \ldots, x_{i-1} = x'_{i-1}$; since $s <^* s'$, one gets $a_l \leqslant a'_l$. Equality is impossible, because this would give $s \sim_k s'$. So $a_l < a'_l$ i.e., $|I_k| < |I'_k|$.

Definition 7.1.12. Assume that D is a qd., then one defines $N(D)$ as follows: if $x \in \underline{h}(D)$, let $s_x = (|I_0|, \ldots, |I_{2n}|)$, with $s \in I_i$ and $n = \mathrm{card}(\mathrm{Occ}(x))$. Then $N(D) = \{s_x; x \in \underline{h}(D)\}$. (One can also introduce ν by $\nu(t) = s_{\varphi(t)}$.)

Theorem 7.1.13. *$N(D)$ is a dendroid.*

Proof. First observe that the function $x \rightsquigarrow s_x$ is strictly increasing. If $x \prec x'$, if $s_x = (|I_0|, \ldots, |I_{2n}|)$, if $s_{x'} = (|I'_0|, \ldots, |I'_{2m}|)$, observe first that $I_{2n} = \{x\}$, $I'_{2m} = \{x'\}$. If $y \in I_{2n}$, and $y \neq x$, then $y \sim_{2n} x$ implies, by Definition 7.1.6(iii), that $\mathrm{Occ}(x)$ has at least $n+1$ elements, a contradiction.

Choose i minimum such that $I_i \neq I'_i$, then $I_i < I'_i$, and, if $i \neq 0$, $I_{i-1} = I'_{i-1}$. By Lemma 7.1.12, $|I_i| < |I'_i|$ (and also $|I_0| = |I'_0|, \ldots, |I_{i-1}| = |I'_{i-1}|$), so $s_x <^* s_{x'}$. This shows that $\underline{h}(N(D)) = \underline{h}(D)$.

Conditions 6.1.1(i) and (ii) are trivially fulfilled. Assume that $x \in \underline{h}(D)$, then the ordinals $|I_1|, \ldots, |I_{2n-1}|$ are pairwise distinct and less than the type of D. If $x < y$, assume that $s_x = (|I_0|, \ldots, |I_{2p}|, a, \ldots)$ and $s_y = (|I_0|, \ldots, |I_{2p}|, b, \ldots)$, and let $s' = \varphi_D^{-1}(x)$, $s'' = \varphi_D^{-1}(y)$, then $s' \sim_{2p} s''$. Choose t as in Definition 7.1.5(ii), then $s' = t * (\underline{c}) * t'$, $s'' = t * (\underline{d}) * t''$, and $|I_1|, \ldots, |I_{2p-1}| \notin [c, d]$. Observe that necessarily $a = c$ and $b = d$. This proves condition 6.1.1(iii).

Condition 6.1.1(iv) is immediate.

Finally, assume that $(|I_0|, \ldots, |I_{n-1}|) \in N(D)^*$ for all n, then $I_0 \supset \cdots \supset I_n \supset \cdots$; as seen in part (i) of the proof of Proposition 7.1.7, the intervals I_{2k+1} are of the form: $\{t_k * s;\ t_k * s \in D\}$, and obviously t_{k+1} is a strict extension of t_k, contradiction with Definition 7.1.1(iv). This proves condition 6.1.1(v).

Proposition 7.1.14. *$N(D)$ is the unique dendroid such that $N(D) \simeq D$.*

Proof. First observe that $N(D) = D$ when D is a dendroid. This is immediate from Remarks 7.1.8 and 7.1.10. Obviously, $N(D)$ and D have the same associated $\underline{h}$, $\sim_k$ and Occ, so $N(D) \simeq D$. Finally, if $D \simeq D'$ and D' is a dendroid, then, since $N(D')$ is built from the data $\underline{h}(D')$, $\sim_k^{D'}$, $\mathrm{Occ}^{D'}$, we get $N(D) = N(D')$; but $N(D') = D'$, so $D' = N(D)$.

Remark 7.1.15. Equivalently: $D \simeq D'$ iff $N(D) = N(D')$.

Definition 7.1.16. If (x, D) and (x', D') are qd., then $I(x, D; x', D')$ is the set of all pairs (f, g) such that:

(i) $f \in I(x, x')$,

(ii) g is a function from D^* to D'^*, which sends D into D',

(iii) $g((\,)) = (\,)$,

(iv) $g(s * (\underline{a})) = g(s) * (\underline{f(a)})$,

(v) if a is an ordinal, $g(s * (a)) = g(s) * (b)$ for some ordinal b (depending on s, a). If $g(s * (a')) = g(s) * (b')$ and $a < a'$, then $b < b'$.

Definition 7.1.17. The following data define a category QDN:
objects: quasi-dendroids (x, D);
morphisms from (x, D) *to* (x', D'): $I(x, D; x', D')$.

Definition 7.1.18. Assume that $(x, D) \simeq (x, D')$, $(y, E) \simeq (y, E')$. If $(f, g) \in I(x, D; y, E)$ and $f', g') \in I(x, D'; y, E')$, then $(f, g) \simeq (f', g')$ means
(i) $f = f'$,
(ii) $\varphi_E g \varphi_D^{-1} = \varphi_{E'} g' \varphi_{D'}^{-1}$,
We shall also use the notation $g \simeq g'$ for $(f, g) \simeq (f, g')$.

Definition 7.1.19. Let D be a qd., and let ν^D be the order preserving map from D to $N(D)$, similarily for D'. If $(f, g) \in I(x, D; x', D')$, then it is possible to define a function $N(g)$ from $N(D)^*$ to $N(D')^*$, in such a way that:

$$\nu^{D'}(g(s)) = N(g)\nu^D(s) \quad \text{for all } s \in D.$$

The functor $F(x, D) = (x, N(D))$, $F(f, g) = (f, N(g))$ is called the functor *normalization*: it is a functor from QDN to DEN.

Proposition 7.1.20. (i) *It is possible to define* $N(g)$ *as in Definition* 7.1.19.
(ii) $g \simeq g'$ *iff* $N(g) = N(g')$.

Proof. Obvious, left to the reader. Observe that, if one defines $\bar{g}(I_k)$ to be the interval J_k (in D') containing $g(I_k)$, then it is immediate that $\bar{g}$ is strictly increasing for $<$, and that $|\bar{g}(I_k)| = f(|I_k|)$ when k is odd. Obviously

$$N(g)(|I_0|, \ldots, |I_{2n}|) = (|\bar{g}(I_0)|, \ldots, |\bar{g}(I_{2n})|).$$

Remark 7.1.21. The notion of quasi-dendroid permits to distinguish between various notions of mutilation:
(i) if (x', D') is a qd. and $f \in I(x, x')$, then it is possible to define $f_{D'}$ and $\mu_f^{D'}$ such that $(f, \mu_f^{D'}) \in I(x, {}^fD'; x', D')$, by:

$$\mu_f(\ldots, x_i, \ldots, \underline{x_j}, \ldots) = (\ldots, x_i, \ldots, \underline{f(x_j)}, \ldots),$$
$${}^fD' = \{s; \mu_f(s) \in D'\}.$$

This corresponds exactly with Example 6.1.5(i). Observe that, even if S' is a dendroid, then ${}^fD'$ needs not to be a dendroid.
(ii) if (x', D') is a qd., if $f \in I(x, x')$, then one defines $f^{-1}(D')$ and $m_f^{D'}$ such that $(f, m_f^{D'}) \in I(x, f^{-1}(D'); x', D')$, by:

$$f^{-1}(D') = N({}^fD'), \qquad m_f^{D'} = N(\mu_f^{D'}).$$

Again this corresponds to Example 6.1.5(ii) and (iii) in the case of a dendroid.
(iii) What makes definition (ii) possible (or at least of some interest) is the compatibility of the mutilation defined in (i) and the equivalence of qd. If $D \simeq D'$, then ${}^fD \simeq {}^fD'$ and $\mu_f^D \simeq \mu_f^{D'}$.

7.2. Constructions involving dendroids

Most of the proofs of this section are omitted.

Definition 7.2.1. (i) Assume that $(D_i)_{i<a}$ is a family of qd of the same type x, then one defines a new qd. $D = \sum_{i<a} D_i$, of type x.

$$D = \{(i) * s;\ s \in D_i\}.$$

Obviously $\underline{h}(\sum_{i<a} D_i) = \sum_{i<a} \underline{h}(D_i)$.

(ii) Assume that $(D'_j)_{j<b}$ is another family of qd. of the same type y, that $h \in I(a, b)$ and $(f, g_i) \in I(x, D_i; y, D'_{h(i)})$ for all $i < a$, then one defines a function $g = \sum_{i<f} g_i$ by:

$$g((i) * s) = (h(i)) * g_i(s).$$

If $D' = \sum_{j<b} D'_j$, then $(f, g) \in I(x, D; y, D')$.

Proposition 7.2.2. (i) *If for all i, $D_i \approx E_i$, then $\sum_{i<a} D_i \approx \sum_{i<a} E_i$. If for all i, $g_i \approx h_i$, then $\sum_{i<f} g_i \approx \sum_{i<f} h_i$.*

If $(D_i)_{i<a}$ is a family of dendroids of the same type, then the notation $\sum_{i<a} D_i$ will denote in fact $N(\sum_{i<a} D_i)$, similarily for $\sum_{i<f} g_i$.

(ii) *If $(D_i)_{i<a}$ is a faimily of sh. dendroids, then the sum $\sum_{i<a} D_i$ is again a sh. dendroid. If $g_i \in I_{\mathrm{sh}}(D_i, D'_{h(i)})$ for all $i < a$, then*

$$\sum_{i<f} g_i \in I_{\mathrm{sh}}\Big(\sum_{i<a} D_i, \sum_{i<b} D'_i\Big).$$

(iii) *The sum of sh. dendroids corresponds to the sum of dilators:*

$$\mathrm{LIN}\Big(\sum_{i<a} D_i\Big) = \sum_{i<a} \mathrm{LIN}(D_i), \qquad \mathrm{LIN}\Big(\sum_{i<f} g_i\Big) = \sum_{i<f} \mathrm{LIN}(g_i).$$

Remarks 7.2.3. One can transfer to dendroids and qd. most of the definitions connected with sums:

(i) a qd. D is *perfect* iff $D \approx D_1 + D_2 \to D_1 = \emptyset$ or $D_2 = \emptyset$, and $\underline{h}(D) \neq 0$. Equivalently, either $\underline{h}(D) = 1$, or $\underline{h}(D) \geq 1$ and there exists s such that all $s' \in D$ can be written $s' = s * (\underline{a}) * t'$ for some a and t', and no element in s is underlined. If D is a dendroid, then $s = (0)$;

(ii) a dendroid D is equal to a sum of perfect dendroids; this decomposition is unique;

(iii) so dendroids can be classified in four kinds, exactly as in Definition 3.1.7. The classification extends to qd. by means of $\approx$.

(iv) of course perfect sh. dendroids correspond to perfect dilators, and sh. dendroids of kinds 0, 1, ω, and Ω correspond respectively to dilators of kinds 0, 1, ω, Ω.

Definition 7.2.4. (i) Assume that $(D_i)_{i<a}$ is a family of qd. of type a, then one

defines a new qd. $D = \sum^*_{i<a} D_i$ of type a;

$$D = \{(\underline{i}) * s;\ s \in D_i\}.$$

(ii) If $(D'_i)_{i<b}$ is another family of qd. of type b, and $h \in I(a, b)$, then, if $(h, g_i) \in I(a, D_i; b, D'_{h(i)})$, one defines $g = \sum^*_{i<h} g_i$, $g((\underline{i}) * s) = (\underline{h}(\underline{i})) * g_i(s)$. Clearly $(h, g) \in I(a, D; b, D')$.

Proposition 7.2.5. (i) *If D is as in Definition* 7.2.4(i), *then D is perfect.*

(ii) *$\sum^*$ is compatible with $\approx$. So, we shall use the notation $\sum^*_{i<a} D_i$, when D_i are dendroids to denote $N(\sum^*_{i<a} D_i)$.*

Remarks 7.2.6. (i) Every qd. can be obtained (modulo $\approx$) by a transfinite iteration of $\sum^*$ and $\sum$.

(ii) This induces the following principle of proof: (*induction on quasi-dendroids*). Assume that $P(\cdot)$ is a property of quasi-dendroids of type x which is compatible with $\approx$, i.e., $P(D)$ and $D \approx D' \to P(D')$, and suppose that

$-P(0)$ (0 is the void dendroid),

$-P(1)$ (1 is a qd. of the form $\{(a)\}$ for some $a \in \mathrm{ON}$);

$-$if for all $i < z$ $P(D_i)$, then $P(\sum_{i<z} D_i)$;

$-$if for all $i < x$ $P(D_i)$, then $P(\sum^*_{i<x} D_i)$.

Then, one can conclude to: for all D of type x, $P(D)$. (Proof: if $\neg P(D)$; if $s \notin D$, let $D_s = \{t; s * (t) \in D\}$, if $s \in D$, let $D = 1$. By hypothesis, $\neg P(D_{(\,)})$. Assume that we have constructed $s_n \in D^* - D$ such that $\neg P(D_{s_n})$. Since (with $s = s_n$) $D \approx \sum_{i<z} D_{s*(i)}$ for some z, or $D \approx \sum^*_{i<x} D_{s*(\underline{i})}$, then in both cases it will be possible to find $s_{n+1} = s_n * (i)$ or $s_n * (\underline{i})$ such that $\neg P(D_{s_{n+1}})$. $P(0)$ and $P(1)$ force s_{n+1} to belong to D^*, but not to D. We have a obtained a s.d.s. in D^*. (N.B. the principle of induction on quasi-dendroids has *nothing to do* with induction on dilators.)

(iii) Assume that D is a sh. dendroid, and let $D_n = (E_n^{-1})(D)$. D_n can be considered as a dendroid of type ω, then $D' = \sum^*_n D_n$ is a sh. dendroid, denoted by $\int D$. Similarly, if $g \in I_{\mathrm{sh}}(D, D')$, define $g_n = g^0(n)$, and $g' = \sum^*_n g_n$, then $g' = \int g \in I_{\mathrm{sh}}(\int D, \int D')$.

(iv) The operations $\int$ on sh. dendroids and on dilators are isomorphic:

$$\mathrm{LIN}\Big(\int D\Big) = \int \mathrm{LIN}(D), \qquad \mathrm{LIN}\Big(\int g\Big) = \int \mathrm{LIN}(g).$$

Remark 7.2.7. It is possible to find the analogue of flowers: a qd. D is a *flower* iff:

(i) $s * (\underline{a}) \in D^*$, $s' * (a') \in D^*$ and s, s' without underlinings $\to s = s'$.

(ii) $s * (\underline{a}) * (s') * (\underline{b}) \in D^*$ and s without underlinings $\to b \geq a$.

(iii) $s, s' \in D$, s without underlinings, s' with underlinings $\to s <^* s'$.

Observe that a sh. dendroid D is a flower iff $\mathrm{LIN}(D)$ is a flower. (Use the fact that F is a flower iff $F = \underline{a} + F'$, with F' perfect and all notations $(z; x_0, \ldots, x_{n-1}; x)_{F'}$ are such that $\sigma_{z,n}(0) = n - 1$).

Definition 7.2.8. (i) define the category ΩSHD:

objects: sh. dendroids of kind Ω, i.e., $D = D' + D''$, D' perfect, $h(D') \neq 1$,

morphisms from D *to* E: the set $\Omega I_{sh}(D, E)$ of $g \in I_{sh}(D, E)$, which can be written $g = g' + g''$, $g' \in I_{sh}(D', E')$, $g'' \in I_{sh}(D'', E'')$, D'', E'' perfect.

(ii) If y is an ordinal, then one can define a functor $\mathrm{SEP}(\cdot)^y$ from ΩSHD to SHD, as follows:

–if D is perfect, let $D_1 = D^0(y+\omega)$. In D_1 take a sequence $(0, \underline{x}) * s' = s$;

(1) if $x \geqslant y$, remove s,

(2) if $x < y$, and some $y' \geqslant x + \omega$ is such that y' occurs in s', remove s,

(3) otherwise, replace s by $s^{\#}$: s can be written

$$s = s_0 * (\underline{a}_0) * \cdots * s_{n-1} * (\underline{a}_n) * s_n, \quad \text{with } s_0, \ldots, s_n$$

without underlinings. Let $s^{\#} = s_0 * t_0 * \cdots * s_{n-1} * t_n * s_n$, with $t_i = (a_i)$ if $a_i \leqslant x$, and $t_i = (x+1, \underline{a_i - x - 1})$ otherwise.

The result of the operations (1)–(3) is D_2. Let $\mathrm{SEP}(D)^y = N(D_2)$.

–if D is not perfect, $D = D' + D''$, let $\mathrm{SEP}(D)^y = D' + \mathrm{SEP}(D'')^y$.

–if $f \in \Omega I_{sh}(D, E)$, then the definition of $\mathrm{SEP}(f)^y$ is left to the reader.

Proposition 7.2.9. *The functors* $\mathrm{SEP}(\cdot)^y$ *on* ΩDIL *and* ΩSHD *correspond to each other:*

$$\mathrm{LIN}(\mathrm{SEP}(D)^y) = \mathrm{SEP}(\mathrm{LIN}(D))^y, \qquad \mathrm{LIN}(\mathrm{SEP}(f)^y) = \mathrm{SEP}(\mathrm{LIN}(f))^y.$$

Definition 7.2.10. (i) Assume that D is a qd. of type x and height a, that E is a qd. of type y and height b and that $b = x$, then one defines a new qd. DE of type y and height a, as follows: given a sequence

$$s = s_0 * (\underline{a}_0) * \cdots * s_{n-1} * (\underline{a}_n) * s_n, \quad \text{with } s_0, \ldots, s_n$$

without underlinings, let $t_i = \varphi_E^{-1}(a_i)$, where φ_E is the order-preserving bijection from E to $\underline{h}(E) = x$. Then define

$$s' = s_0 * t_0 * \cdots * s_{n-1} * t_n * s_n$$

and let $DE = \{s'; s \in D\}$.

(ii) If D and E are dendroids, then DE will denote $N(DE)$.

(iii) If D and E are sh. dendroids, then one defines $D \circ E$ by: $D \circ E = D^0(\underline{h}(E))E$; $D \circ E$ is a sh. dendroid (this follows from Proposition 7.2.11 below).

Proposition 7.2.11. (i) *Assume that* $(f, f') \in I(x, D; x', D')$, $(g, g') \in I(y, E; y', E')$, *that* $x = \underline{h}(E)$, $f = \underline{h}(g')$, *then one can define* $(g, f'') \in I(y, DE; y', D'E')$ *by the condition* $\underline{h}(f'') = \underline{h}(f')$.

(ii) *Hence* ${}^f(DE) = {}^{h(\mu_f^E)}D) \circ {}^fE$, *and*

$$\mu_f^{DE} = \mu_h^D(\mu_f^E) \circ \mu_f^E.$$

(iii) *If* $f \in I_{sh}(D, D')$, $g \in I_{sh}(E, E')$, *then one can define* $f \circ g \in I_{sh}(D \circ E, D' \circ E')$ *by:* $f \circ g = D^0(g)$.

(iv) *The functor* $\circ$ *defined by Definition* 7.2.10 *and Proposition* 7.2.11(iii) *corresponds to* $\circ$ *in* DIL:

$$\mathrm{LIN}(D\circ D')=\mathrm{LIN}(D)\circ\mathrm{LIN}(D'),\qquad \mathrm{LIN}(f\circ f')=\mathrm{LIN}(f)\circ\mathrm{LIN}(f').$$

7.3. *Hierarchies*

Definition 7.3.1. Assume that D is a qd of type ω, and that D is *finitary*: this means that, for all $s\in D^*$, the set $\{a;\ a\in\mathrm{ON}$ and $s*(a)\in D^*\}$ is finite. Then one defines the following hierarchies of number-theoretic functions:

(i)

$$\gamma_0(n)=0,\qquad \text{where } 0 \text{ is the void qd.}$$
$$\gamma_1(n)=1,\qquad \text{where } 1 \text{ is any qd. } D=\{(a_0)\},\quad \text{for some } a_0\in\mathrm{ON}.$$
$$\gamma_{\Sigma_{i<k}D_i}(n)=\gamma_{D_0}(n)+\cdots+\gamma_{D_{k-1}}(n).$$
$$\gamma_{\Sigma_{i<\omega}D_i}(n)=\gamma_{D_0}(n)+\cdots+\gamma_{D_{n-1}}(n).$$

γ is called the *pointwise hierarchy*. The name comes from the fact that $\gamma_D(n)$ can be computed from values $\gamma_{D'}(n)$, with the same n, or equivalently, what is defined by induction is the value $\gamma_D(n)$ (and not the function γ_D)

(ii)

$$\delta_0(m,n)=0.$$
$$\delta_1(m,n)=1.$$
$$\delta_{\Sigma_{i<k+1}D_i}(m,n)=\gamma_{D_0}(m)+\cdots+\gamma_{D_{k-1}}(m)+\delta_{D_k}(m,n),\quad \text{if } \underline{h}(D_k)\neq 0.$$
$$\delta_{\Sigma^*_{i<\omega}D_i}(m,n)=\gamma_{D_0}(m+1)+\cdots+\gamma_{D_{n-1}}(m+n).$$

δ is called the *double pointwise hierarchy*.

(iii)

$$\theta_0(m,n)=n.$$
$$\theta_1(m,n)=m+n.$$
$$\theta_{\Sigma_{i<k}D_i}(m,n)=\theta_{D_0}(m,\theta_{D_1}(m,\ldots\theta_{D_{k-1}}(m,n)\ldots)).$$
$$\theta_{\Sigma^*_{i<\omega}D_i}(m,n)=\theta_{D_0}(m,\theta_{D_1}(m,\ldots\theta_{D_{n-1}}(m,n)\ldots)).$$

(iv) In general, if f is a function from N^2 to N, define $g=\mathrm{Un}(f)$, as follows:

$$g(n)=f(n,0)+(f(n+1,1)-f(n-1,0))+(f(n-2,2)-f(n-2,1))+\cdots$$
$$+(f(0,n)-f(0,n-1)).$$

It is easy to see that $\gamma_D=\mathrm{Un}(\delta_D)$; one defines $\lambda_D=\mathrm{Un}(\theta_D)$. λ_D is the *iterative hierarchy*, whereas θ_D is the *double iterative hierarchy*.

Proposition 7.3.2. (i) *If* $D\simeq D'$, *then* $\gamma_D=\gamma_{D'}$, $\delta_D=\delta_{D'}$, $\theta_D=\theta_{D'}$, $\lambda_D=\lambda_{D'}$.

(ii) $\gamma_D(n)=\underline{h}(E_n^{-1}(D))$.

(iii) *if D is a sh. dendroid of kind Ω, then*

$$\delta_D(m,n)=\gamma_{D'}(m), \quad \textit{with } D'=\mathrm{SEP}(D)^n,$$

(iv) *θ and λ are related to $\bigwedge$ in the following way: if $D=\mathrm{BCH}(F)$, then (provided $\mathrm{BCH}(\bigwedge D)$ is finitary)*

$$\lambda_D(m)=(\bigwedge F)(m), \qquad \theta_D(m,n)=(\bigwedge F)(m,n).$$

Proof. (i) Trivial. In fact, when we define the hierarchies by

$$\gamma_{\sum_{i<k}D_i}(n)=\ldots,$$

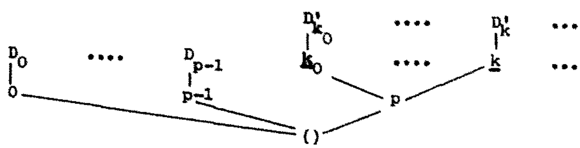
we are already working modulo $\approx$.

(ii) By induction on D. If D is void, then $E_n^{-1}(D)$ is void, and $\gamma_D(n)=0$. If $D=\{(a_0)\}$ for some $a_0\in\mathrm{ON}$, then $E_n^{-1}(D)=D$, and $\gamma_D(n)=\underline{h}(D)=1$. If $D=\sum_{i<k}D_i$, then

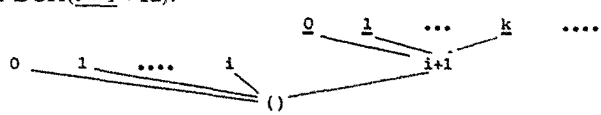
$$E_n^{-1}(D)=\sum_{i<k}E_n^{-1}(D_i) \quad\text{and}\quad \underline{h}(E_n^{-1}(D))=\sum_{i<k}\underline{h}(E_n^{-1}(D_i)).$$

The induction hypothesis gives the result. Similarily, if $D=\sum^*_{i<\omega}D_i$, then $\underline{h}(E_n^{-1}(D))=\sum_{i<n}\underline{h}(E_n^{-1}(D_i))$, etc...

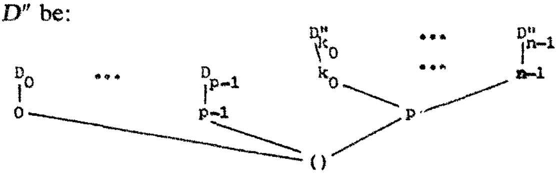
(iii) Assume that D is a sh. dendroid of kind Ω, then D can be written



where $D_0,\ldots,D_{p-1},D'_{k_0},\ldots,D'_k,\ldots$ are qd. Define $D''_i=D'_iT_i$, where T_i is the dendroid $\mathrm{BCH}(\underline{i+1}+\mathrm{Id})$:



and let D'' be:



Then it is immediate that $D''\approx\mathrm{SEP}(D)^n=D'$. Also, $\gamma_{D''}(m)$ is equal to

$$\gamma_{D_0}(m)+\cdots+\gamma_{D_{p-1}}(m)+\gamma_{D'_k}(1+m)+\cdots+\delta_{D'_{n-1}}(n+m)=\delta_D(m,n).$$

(iv) We first prove a lemma: if D is a qd. of kind Ω, and if $D'=D\circ T_0$ (T_0 as in (iii) above), then $\theta_{D'}(m,n)=\theta_D(m,1+n)-1$. We argue by induction on D: the cases 0 and 1 are immediate, since $D' = D$ in these cases. If $D = \sum_{i<k} D_i$, then $D' = \sum_{i<k} D_i'$, and

$$\theta_{D'}(m, n) = \theta_{D_0'}(m, \ldots \theta_{D_{k-1}'}(m, n)) = \theta_{D_0}(m, \ldots \theta_{D_{k-1}}(m, n+1) \ldots) - 1,$$

using the induction hypothesis. If $D = \sum^*_{i<\omega} D_i$, then $D' = D_0' + \sum^*_{i<\omega} D_{i+1}'$, and

$$\begin{aligned}\theta_{D'}(m, n) &= \theta_{D_0'}(m, \ldots \theta_{D_n'}(m, n) \ldots)) \\ &= \theta_{D_0}(m, \ldots \theta_{D_n}(m, n+1) \ldots) - 1 = \theta_D(m, n+1) - 1.\end{aligned}$$

Now, we prove (iv), by induction of F such that $D = \mathrm{BCH}(F)$. Observe that F is weakly finite. If $F = \underline{0}$ or $F = \underline{1}$, then this is immediate. If $F = F' + \underline{1}$, then $\theta_D(m, n) = \theta_{D'}(m, m+n)$, and $(\bigwedge F)(m, n) = (\bigwedge F')(m, m+n)$. If F is of kind Ω, write $\mathrm{SEP}(F)^n = G + F_0 + \cdots + F_{n-1}$, then we have:

$$(\bigwedge F)(m, n) = (\bigwedge F_0)(m, \ldots 1 + (\bigwedge F_{n-1})(m, 0) \ldots).$$

Let $E = \mathrm{BCH}(G)$, $D_i''' = \mathrm{BCH}(F_i)$, then, with the notations of (iii) above, $E = D_0 + \cdots + D_{p-1}$; $D_i''' = D_i''$ if $i \geq k_0$, $D''' = 0$ otherwise. Hence, using the induction hypothesis:

$$(\bigwedge F)(m, n) = \theta_E(m, 1 + \theta_{D_0''}(m, \ldots 1 + \theta_{D_{n-1}''}(m, 0) \ldots)$$

But, by the lemma just proved, $\theta_{D_i'}(m, p) = \theta_{D_i'}(m, p+i+1) - i - 1$; hence

$$(\bigwedge F)(m, n) = \theta_E(m, \theta_{D_0}(m, \ldots \theta_{D_{n-1}}(m, n) \ldots) = \theta_D(m, n).$$

The equation $\lambda_D(m) = (\bigwedge F)(m)$ can easily be obtained by applying Un to both sides, remarking that, if $f(n, m) = F(n, m)$, then $\mathrm{Un}(f)(m) = \mathrm{Un}(F)(m)$.

Theorem 7.3.3. (comparison of hierarchies). *If D is a sh. dendroid, define $D' = \bigwedge D$ by: $D' = \mathrm{BCH}(\bigwedge \mathrm{LIN}(D))$; then (provided D and D' are finitary)*

(i) $\theta_D(m, n) = \delta_{\bigwedge D}(m, n)$,

(ii) $\lambda_D(m) = \gamma_{\bigwedge D}(m)$,

for all $m, n < \omega$.

Proof. (i) $\theta_D(m, n) = (\bigwedge F)(m, n) = \delta_{\bigwedge D}(m, n)$, with $F = \mathrm{LIN}(\Gamma)$.

(ii) comes from (i) by applying Un to both sides.

Remarks 7.3.4. (i) For the 'natural' choice of D such that $\underline{h}(D) = \varepsilon_0$, one gets $\underline{h}(\bigwedge D) =$ the Howard ordinal (see [10, 18]). So we obtain the most interesting particular case of Theorem 7.3.3, incorrectly stated:

$$\theta_{\varepsilon_0} = \delta_H$$

($H =$ Howard ordinal).

(ii) In Theorem 7.3.3, the restriction that D and D' are finitary can be removed by defining the hierarchies on arbitrary dendroids. Then the theorem can be stated: if one side is finite, so is the other, and it is equal. Another solution consists in defining a property of sh. dendroids (to be finitistic) that implies finitarism, and that is preserved by: in that case the hypothesis of Theorem 7.3.3 will be 'D finitistic'. This will be done by means of multi-dendroids.

7.4. *Multi-quasi-dendroids*

Definition 7.4.1. Let K be a finite or denumerable set (the set of *colours*), and let $(x_k)_{k\in K}$ be a family of ordinals. A *multi-quasi-dendroid* (mqd.) of type (x_k) is a pair $((x_k), D)$, such that D is a set of finite sequences $(a_0, \ldots, a_n)$, and:

(i) if $(a_0, \ldots, a_n) \in D$ and $m < n$, then $(a_0, \ldots, a_m) \notin D$;

(ii) if $(a_0, \ldots, a_n) \in D$, then, for all $i \leq n$, a_i is either an ordinal or a pair (x, k), x ordinal, $k \in K$. Such a pair will be written $\underset{k}{x}$, and one will say that $\underset{k}{x}$ is *of colour k* (the ordinal x is said to be *white*);

(iii) if $s*(t)$ and $s*(u)$ belong to D^*, and if t is of colour k, then u is of colour k (hence, if t is white, so is u);

(iv) there is no sequence (a_n) such that, for all n, $(a_0, \ldots, a_{n-1}) \in D^*$.

Definition 7.4.2. A mqd. $((x_k), D)$ is said to be a *multi-dendroid* iff it enjoys the extra properties:

(v) if $(a_0, \ldots, a_n) \in D$, then n is even;

(vi) if $(a_0, \ldots, a_n) \in D$, if $2i \leq n$, then a_{2i} is white. If $2i+1 < n$, then a_{2i+1} is coloured;

(vii) if $s*(x) \in D^*$, x ordinal, and $x' < x$, then $s*(x') \in D^*$,

(viii) if

$$(a_0, \ldots, a_{2i}, \underset{k}{x}) \quad \text{and} \quad (a_0, \ldots, a_{2i}, \underset{k}{x'}) \in D^*,$$

and if for some $j < 2i$, $a_j = \underset{k}{x''}$, then $x'' \notin [x, x']$.

Remarks 7.4.3. (i) Allowing more than one colour corresponds to the idea of having many variables, for instance, if one calls a K-dilator a functor from ON^K (the product of K copies of ON) to ON commuting to $\varinjlim$ and &, then it will be possible to prove an isomorphism theorem between K-dilators and sh. homogeneous (w.r.t. all colours) multi-dendroids coloured with colours from K. In practice, this will be done for bilators.

(ii) But the most interesting aspect of quasi-dendroids is that they permit an approach to Π^1_2-logic completely free from categories: usual dendroids, when they are strongly homogeneous, permit to represent dilators, but it is still necessary to consider morphisms of sh. dendroids, and these morphisms are by no way encoded by the tree structure. Let us explain how it will work in the case of multi-dendroids. A typical situation is when the multi-dendroid will be homogeneous w.r.t. the colour 0, but not w.r.t. the remaining colours in $K' = K - \{0\}$, then the mutilation functions corresponding to the colours in K' induce natural transformations of the functors constructed by considering only the colour 0.

(iii) In practice, colours appear naturally in the process of considering predecessors of a given sh. dendroid. Start, say with a sh. dendroid D of kind Ω (one colour); SEP(D) is bichromatic (a 'two-variables' dendroid) and homogeneous w.r.t. the two colours; but the predecessors SEP$(D)^y$ are no longer homogeneous w.r.t. y, but the branchings of colour, say 1, permit to encode the natural transformations SEP$(D)^f$ ... If, for some y, SEP$(D)^y$ is of kind Ω, then we need a third colour in order to construct its predecessors.

(iv) We define $\underline{t}((x_k), D) = \underline{t}(D) = (x_k)$, and, for $l \in K$, $\underline{t}_l((x_k), D) = \underline{t}_l(D) = x_l$; $\underline{h}(D)$ is defined as usual.

(v) For notational purposes, we shall use (if 0 and 1 are two colours), $\underline{x}$ instead of $\underset{0}{x}$, and $\underline{\underline{x}}$ for $\underset{1}{x}$.

Definition 7.4.4. (i) If D is a mqd. and if $s \in D$, one defines Occ(s) by:
–in s remove all white elements,
–then remove all points a_i such that for some $j < i$, $a_j \in s$.

The resulting sequence is Occ(s). For instance, if $s = (0, \underline{\underline{1}}, \underline{2}, \omega, \underline{1}, 3, \underline{\underline{5}}, \omega, \underline{5})$, then Occ$(s) = (\underline{\underline{1}}, \underline{2}, \underline{1}, \underline{\underline{5}})$.

(ii) We define OCC$^D(z)$, when $z < \underline{h}(D)$ by: Occ$(s) =$ Occ$^D(\varphi_D(z))$ (φ_D is the order-preserving isomorphism from D to $\underline{h}(D)$).

Definition 7.4.5. We define the equivalence relation $\approx$ between mqd. of the same type: $D \approx D'$ iff

(i) $\underline{h}(D) = \underline{h}(D')$,

(ii) for all $z \in h(D)$, Occ$^D(z) =$ Occ$^{D'}(z)$,

(iii) let $k \in K$, and $E = k^*(D)$, $E' = k^*(D')$, then the equivalence relations $\sim_n^E$ and $\sim_n^{E'}$, defined on $\underline{h}(E) = \underline{h}(D) = \underline{h}(D') = \underline{h}(E')$ coincide.

Theorem 7.4.6. *Every equivalence class modulo $\approx$ contains one and only one multi-dendroid.*

Proof. Theorem 7.4.6 is a generalization of the main result of Section 7.1. We give just a sketch of the proof: the main idea is to define a multi-dendroid $N(D)$ such that $N(D) \approx D$, where D is a given mqd. If $x \in \underline{h}(D)$, define $s_x = (a_0, \ldots, a_{2n})$, with:

(i) $(a_1, a_3, \ldots, a_{2n-1}) =$ Occ$^D(x)$,

(ii) $a_0 =$ order type of $\{I'_0; I'_0 < I_0\}$ (if $x \in I_0$),

(iii) $a_{2p+2} =$ order type of $\{I'_{2p+2}; I'_{2p+2} < I_{2p+2}$ and $I'_{2p+2} \subset I_{2p+1}\}$ if $x \in I_{2p+2} \subset I_{2p+1}$). (the intervals I_p used here are equivalence classes modulo $\sim_p$, where

(1) $s \in D \rightarrow s \sim_p^D s$,

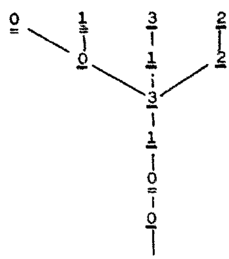
(2) if $s, s' \in D$, and $s \neq s'$, then:
$-s \sim_{2p+1} s'$ if for some t, $s = t * u$, $s' = t * u'$, and Occ(t) has exactly p elements.
$-s \sim_{2p} s'$ if one can find ordinals z, z', a colour $k \in K$, and $t \in D^*$, such that

$$s = t * (\underset{k}{z}) * u, \qquad s' = t * (\underset{k}{z'}) * u',$$

$\mathrm{Occ}(t)$ has exactly p elements, and for all z'' such that $(\underset{k}{z''}) \in \mathrm{Occ}(t)$, $z'' \notin [z, z']$.

The fact that $N(D)$ is a multi-dendroid, with $N(D) \simeq D$, is obvious, as well as the unicity of such a $N(D)$.

Let us conclude with an example:



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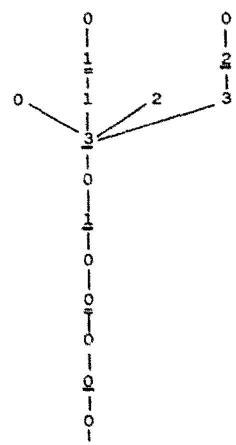


N(D)

Definition 7.4.7. (i) If $((y_k), D)$ is a mqd., if for all $k \in K$, $f_k \in I(x_k, y_k)$, then one defines a mqd. ${}^{(f_k)}((y_k), D) = ((x_k), (f_k)^{-1}(D))$:

$$(\ldots, (\underset{k}{z}), \ldots, z', \ldots) \in {}^{(f_k)}D \quad \text{iff } (\ldots, \underset{k}{(f_k(z))}, \ldots, z', \ldots) \in D;$$

one defines a function $\mu^D_{(f_k)}$ from ${}^{(f_k)}D^*$ to D^*, by:

$$\mu_{(f_k)}(\ldots, (\underset{k}{z}), \ldots, z', \ldots) = (\ldots, \underset{k}{(f_k(z))}, \ldots, z', \ldots).$$

(ii) If $((y_k), D)$ is a multi-dendroid, if $f_k \in I(x_k, y_k)$, then it is possible to define a multi-denroid $(f_k)^{-1}(D)$, by

$$(f_k)^{-1}(D) = N({}^{(f_k)}D).$$

One also defines a mutilation function $m^D_{(f_k)}$ from $(f_k)^{-1}D^*$ to D^*, by the condition:

$$m^D_{(f_k)}(s) = \mu^D_{(f_k)}(\varphi_E^{-1}(\varphi_{N(E)}(s))), \quad \text{for all } s \in N(E)$$

with $E = {}^{(f_k)}D$.

Remark 7.4.8. What makes Definition 7.4.7(ii) possible is the compatibility of the operation ${}^{(f_k)}\cdot$ with the equivalence.

Remark 7.4.9. All constructions of Section 7.2 can be carried out, *mutatis mutandis* in the context of mqd.'s, for instance:

(i) Assume that $(D_i)_{i<a}$ is a family of mqd. of the same type (x_k), then one defines a new mqd. $D = \sum_{i<a} D_i$, of type (x_k):

$$D = \{(i) * s;\ s \in D_i\}.$$

(ii) Assume that $(D_i)_{i<a}$ is a family of mqd. of the same type $(x_k)_{k\in K}$, and that $l \in K$, and $a = x_l$, then one defines a new mqd. $D = \sum_{i<a}^{*l} D_i$, of type (x_k):

$$D = \{(\underset{l}{\underline{i}}) * s;\ s \in D_i\}.$$

(iii) It is possible to classify the mqd. in four classes, but the classification depends on a colour. More precisely, if D is a mqd. of type (x_k) and l is a colour in K, say that D is of kind 0, 1, ω, Ω (w.r.t. l) iff $l^*(D)$ is of kind 0, 1, ω, Ω.

Definition 7.4.10. Let D be a mqd. of type $(x_k)_{k\in K}$; let K' be a subset of K. One says that D is *K'-homogeneous* iff

($*$) given $(y_k)_{k\in K}$, $f_k \in I(y_k, x_k)$, $g_k \in I(y_k, x_k)$, such that for $k \in K - K'$, $y_k = x_k$ and $f_k = g_k = E_{x_k}$, then

$$(f_k)^{-1}(D) = (g_k)^{-1}(D);$$

($**$) under the hypotheses of (i), let $m_{(f_k)}$ and $m_{(g_k)}$ be the mutilation functions from $(f_k)^{-1}(D)^* = (g_k)^{-1}(D)^*$ to D^*, associated with (f_k) and (g_k), as in Definition 7.4.7(ii), then if $s \in N((f_k)^{-1}(D))^*$, and if for all $(\underset{l}{x})$ occuring in s, we have $f_l(x) = g_l(x)$, one has:

$$m_{(f_k)}(s) = m_{(g_k)}(s).$$

Proposition 7.4.11. *If D is a mqd. of type (x_k), and if for all $k \in K$, $f_k \in I(y_k, x_k)$, then D K'-homogeneous $\to (f_k)^{-1}(D)$ k'-homogeneous.*

Proof. Immediate, left to the reader.

Definition 7.4.12. Let D be a mqd. of type (x_k), and let $K' \subset K$; D is *strongly K'-homogeneous* iff for all $k \in K'$, $x_k = \omega$, and if for all family (y_k) with $y_k \geqslant \omega$ for all $k \in K'$ and $y_k = x_k$ for all $k \in K - K'$, one can find a mqd. D' such that D' is K'-homogeneous and $D = (E_{x_k y_k})^{-1}(D')$. Shortly one says that D is a K'-shmqd., when considered as a K'-shmqd, then the type of D is $(x_k)_{k\in K-K'}$.

Definition 7.4.13. (i) Assume that D is a K'-shmqd of type $(x_k)_{k\in K-K'}$, then given any sequence $(z_k)_{k\in K'}$, one defines a multi-dendroid $D^0((z_k))$:

–assume that $z_k \geqslant \omega$ for all $k \in K'$. Let $y_k = z_k$ for $k \in K'$, and $y_k = x_k$ for $k \in K - K'$, then if D' is as in Definition 7.4.12, let $D((z_k)) = N(D')$;

–in general, define $y'_k = z'_k = \sup(z_k, \omega)$ for $k \in K'$, $y'_k = x_k$ otherwise; then one defines $D((z_k)) = (E_{y_k y'_k})^{-1}(D((z'_k)))$.

(ii) Assume that D is a K'-shmqd of type $(x_k)_{k\in K-K'}$, and let $(f_k)_{k\in K'}$ be such that $f_k \in I(u_k, v_k)$ for all $k \in K'$, then one defines a function $D^0((f_k))$ from $D^0((u_k))$ to $D^0((v_k))$ as follows. Let $g_k = f_k$, if $k \in K'$, $g_k = E_{x_k}$, if $k \in K-K'$, then $D^0((f_k)) = m_{(g_k)}^{D^0((v_k))}$.

(iii) Assume that D is a K'-shmqd of type $(y_k)_{k\in K-K'}$ and let $(f_k)_{k\in K-K'}$ be a family of morphisms $f_k \in I(x_k, y_k)$; then define $g_k = f_k$ if $k \in K'-K$, $g_k = E_\omega$ if $k \in K'$, then $(g_k)^{-1}(D)$ is a K'-shmqd of type (x_k), denoted by $(f_k)^{-1}(D)$.

Definition 7.4.14. (i) Assume that D is a K'-shmqd of type $(x_k)_{k\in K-K'}$, then one defines a functor $\mathrm{LIN}^{K'}(D)$ from the category $\mathrm{ON}^{K'}$ to ON:

$-\mathrm{LIN}^{K'}(D)((x_k)) = \underline{h}(D^0((x_k)))$,

$-\mathrm{LIN}^{K'}(D)((f_k)) = \underline{h}(D^0((f_k)))$.

(ii) Assume that D is a K'-shmqd of type $(y_k)_{k\in K-K'}$, and that $(f_k)_{k\in K-K'}$ is such that $f_k \in I(x_k, y_k)$, then one defines a natural transformation $\mathrm{LIN}^{K'}((f_k))$ from $\mathrm{LIN}^{K'}((f_k)^{-1}(D))$ to $\mathrm{LIN}^{K'}(D)$:

$$\mathrm{LIN}^{K'}(f_k)(\omega) = \underline{h}(m_{(g_k)}^{D})$$

where (g_k) is as in Definition 7.4.13(iii). (The main cases are $K' = \{0\}$, $K' = \{1\}$, $K' = \{0, 1\}$; we shall use LIN^0, LIN^1, LIN^{01} for the corresponding $\mathrm{LIN}^{K'}$. We recall that colour 0 is abbreviated into $-$, and colour 1 into $=$.)

7.5. The function $\bigwedge$

Definition 7.5.1. (i) A 0-schmqd is of *kind* Ω iff it can be written $D = D' + \sum_{n<\omega}^{*0} D''_n$, and $\mathrm{LIN}^0(D)$ is of kind Ω.

(ii) A 1-shmqd is a *flower* iff it can be written $D = D' + \sum_{n<\omega}^{*1} D''_n$, and $\mathrm{LIN}^1(D)$ is a flower; the flower D is *regular* iff one can write $D = D' + \sum_{n<\omega}^{*1} (1 + D''_n)$, where $1 = \{(0)\}$.

(iii) A 01-shmqd is a *bilator* iff, as a 1-shmqd, it is a flower, and $\mathrm{LIN}^{01}(D)$ is a bilator. The bilator D is *regular*, iff the 1-shmqd D is a regular flower.

Definition 7.5.2. (i) Assume that D is a 0-shmqd of type $(x_k)_{k\in K'}$ and that $1 \notin K$; then, if D is of kind Ω, one defines a 01-shmqd $\mathrm{SEP}(D)$, which is a bilator, as follows: let $D = D' + \sum_{n<\omega}^{*0} D''_n$; then $\mathrm{SEP}(D) = D' + \sum_{n<\omega}^{*1} C''_n$, where, for each n, C''_n is obtained by replacing in D''_n, in all sequences s, points $\underline{x}$ with $x \leq n$ by $0, \underline{\underline{x}}$, and points $\underline{y}$, with $y > n$, by $1, \underline{y-n-1}$. (For instance, when $n = 5$, the sequence $(0, \underline{7}, \underline{8}, 6, \underline{4})$ becomes $(0, 1, \underline{1}, \underline{8}, 6, 0, \underline{\underline{4}})$).

(ii) Assume that D is a 01-shmqk of type $(x_k)_{k\in K}$, and that D is a bilator, then one defines a 0-shmqd $\mathrm{UN}(D)$, which is of kind Ω, as follows: let $D = D' + \sum_{n<\omega}^{*1} D''_n$, then $\mathrm{UN}(D) = D' + \sum_{n<\omega}^{*0} C''_n$, where, for each n, C''_n is obtained by replacing in D''_n, in all sequences s, points $\underline{\underline{x}}$ by $\underline{x}$, and points $\underline{y}$ by $n+1+y$. (For instance $(0, 1, \underline{1}, \underline{8}, 6, 0, \underline{\underline{4}})$ becomes $(0, 1, \underline{7}, \underline{8}, 6, 0, \underline{4})$).

Theorem 7.5.3. Un *and* SEP *correspond to the operations defined on dilators:*

(i) $\mathrm{SEP}(\mathrm{LIN}^0(D)) = \mathrm{LIN}^{01}(\mathrm{SEP}(D))$

(ii) $\mathrm{SEP}(\mathrm{LIN}^0(\nu^D_{(f_k)})) = \mathrm{LIN}^{01}(\nu^{\mathrm{SEP}(D)}_{(f_k)})$.

Definition 7.5.4. Assume that D and D' are 1-shmqd and regular flowers, then one defines a 1-shmqd $D \circ_1 D'$, which is regular flower, as follows:

(i) let $D_1 = D^0(\underline{h}(D'))$,

(ii) in all sequences of D_1, replace any element $\underline{x}$ by the finite sequence $t = \varphi^{-1}_{D_1'}(x)$.

Theorem 7.5.5. *Assume that D and D' are* 01*-shmqd and are regular bilators; then $D \circ_1 D'$ is* 01*-strongly homogeneous, and is a regular bilator; moreover*

(i) $\mathrm{LIN}^{01}(D) \circ_s \mathrm{LIN}^{01}(D') = \mathrm{LIN}^{01}(D \circ_1 D')$,

(ii) $\mathrm{LIN}^{01}(\nu^D_{(f_k)}) \circ_s \mathrm{LIN}^{01}(\nu^{D'}_{(f_k)}) = \mathrm{LIN}^{01}(\nu^{D \circ_1 D'}_{(f_k)})$.

Theorem 7.5.6. *Assume that $(D_i)_{i<x_w}$ is a family of* 1*-shmqd of type $(x_k)_{k\in K}$ which are regular flowers; then one defines a* 1*-shmqd $\prod^{*w}_{i<x_w} D_i$ of type $(x_k)_{k\in K}$, which is a regular flower, and such that:*

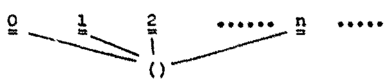
(i) *If, for all $i < x_w$, D_i is a* 01*-shmqd and is a regular bilator, then the product $\prod^{*w}_{i<x_w} D_i$ is a regular bilator and:*

$$\prod_{i<x_w} \mathrm{LIN}^{01}(D_i) = \mathrm{LIN}^{01}\Big(\prod_{i<x_w}^{*w} D_i\Big).$$

(ii) Similarily, if $f_k \in I(x_k, y_k)$,

$$\prod_{i<f_w} \mathrm{LIN}^{01}(\nu^{D_i}_{(f_k)}) = \mathrm{LIN}^{01}(\nu^{\prod^{*w}_{i<x_w} D_i}_{(f_k)}).$$

Proof. One constructs $\prod^{*w}_{i<x_w} D_i$ by introducing the partial products $\prod^{*w}_{x'\leq i<x} D_i$, for $x \leq x_w$, and these partial products are built by induction on x: if $x = x'$, then $\prod^{*w}_{x'\leq i<x} D_i$ is simply the mqd.



If $x = x'+1$, then write $D_{x'} = D' + \sum^{*1}_{n<\omega}(1 + D''_n)$ and let:

$$C' = \{(\underset{w}{\underline{x}'}) * s;\ s < D'\}, \qquad C''_n = \{(\underset{w}{\underline{x}'}) * s;\ s \in D''_n\}.$$

Then define

$$\prod^{*w}_{x'\leq i<x} D_i = C' + \sum^{*1}_{n<w}(1 + C''_n).$$

If $x > x'$, and x limit (one considers only the case x denumerable, which is the only one which matters), write $x = \sup(x_n)$, where x_n is an increasing sequence, and $x_0 = x'$; by induction hypothesis, the partial products $P_n = \prod^{*w}_{x_n\leq i<x_{n+1}}$ have

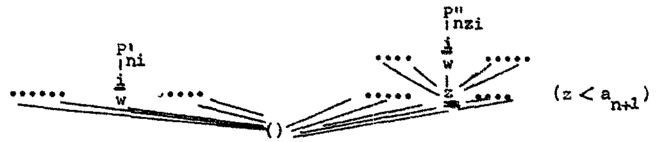
already been constructed. These products can be written $P_n = P'_n + \sum^{*1}_{m<w}(1+P''_{nm})$. We shall use the extra induction hypothesis:

$$\text{(H)}\qquad P'_n = \sum^{*w}_{x_n \le i < x_{n+1}} P'_{ni}, \qquad P''_{nm} = \sum^{*w}_{x_n \le i < x_{n+1}} P''_{nmi}.$$

(1) Assume that $F_n = \mathrm{LIN}^1(P_n)$, and assume that f enumerates the intersection of the ranges of the functions $x \rightsquigarrow F_n(x)$; fix an ordinal a, and define $a_0 = f(a)$, a_{n+1} by $f(a) = F_0(\ldots(F_n(a_{n+1}))\ldots)$. Starting with the mqd.

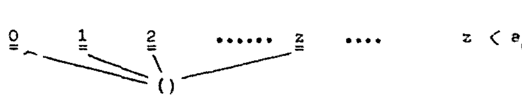
$$P^0_n(a_{n+1}) = P'_n + \sum^{*1}_{z<a_{n+1}}(1+P''_{nz}),$$

one defines the tree T^a_n:



Observe that the ground level of this tree is not homogeneous in colour. This violates Definition 7.4.1(iii). Moreover, what we have written represents only the sequences of T^a_n of maximal length: the sequences $(\underline{\underline{f_n(z)}})$ (for $z<a, f_n(z)$ is defined by: $f(z) = F_0(\ldots(F_n(f_n(z)))\ldots)$) are also in T^a_n by definition, in violation of Definition 7.4.1(i). The tree T^a_n is ordered as usual between maximal sequences, and by $(\underline{\underline{f_n}}(z)) < (\underline{\underline{f_n}}(z)) * s$, when $s \ne (\ )$: this order is opposite to the familiar one in that case. The order type of T^a_n is equal to $F_n(a_{n+1}) = a_n$. Let φ^a_n be the order-preserving map from T^a_n to a_n.

(2) By induction on n, we construct a tree C^a_n: C^a_0 is



If C^a_n has been obtained, replace in C^a_n all points $\underline{\underline{z}}$ occurring in the sequences by the finite sequences $(\varphi^a_n)^{-1}(z)$: the result is C^a_{n+1}.

(iii) It is immediate that, given any point $(\underline{\underline{z}})$ in C^a_0, it has only finitely many distinct successive avatars s^z_n in the C^a_n's, hence, for sufficiently great integers n, $s^z_n = s^z =$ constant. We define the tree C^a_ω to be: $C^a_\omega = \{s^z : z < a_0\}$.

(iv) The only ordinals of colour 1 in C^a_ω are the points $\underline{\underline{u(z)}}$ defined by: $f_n(u(z)) = f(z)$ for sufficiently great n. Define D^a_ω by replacing in C^a_ω all points $\underline{\underline{u(z)}}$ by $\underline{\underline{z}}$; we have now to make a mqd. with D^a_ω.

(v) It is easy to see that all levels in D^a_ω are correctly made, except some levels which are of colour w on the left, of colour 1 on the right: so replace in D^a_ω all points $\underset{w}{z}$ by 0, $\underset{w}{z}$, and all points $\underline{\underline{z}}$ by 1, $\underline{\underline{z}}$: the resulting tree enjoys Definition 7.4.1(iii): let us call it D'^a_ω.

(vi) In D'^a_ω, replace any sequence $(b_0, \ldots, b_n)$ by $(b_0, 1, b_1, 1, \ldots, 1, b_n, 0)$, then the resulting tree D''^a_ω enjoys Definition 7.4.1(i): D''^a_ω is a mqd.

We define $P = \prod_{x' \leqslant i < x} D_i = D''^{\omega}_{\omega}$; if $f_1 \in I(a, b)$, then ${}^{[f_1]}D''^{b} = D''^{a}$: this permits to verify strong homogeneity of P. The extra hypothesis (H) is verified by P: one can write

$$P = P' + \sum_{m<\omega}^{*1} (1 + {}''_{m}), \quad \text{with } P' = \sum_{x' \leqslant i < x}^{*w} P'_i, \quad P''_m = \sum_{x' \leqslant i < x}^{*w} P''_{mi}.$$

Theorem 7.5.7. *Assume that $(D_i)_{i<x}$ is a family of 1-shmqd of type $(x_k)_{k \in K}$, which are regular flowers; then one defines a 1-shmqd $\prod_{i<x} D_i$, which is a regular flower of type $(x_k)_{k \in K}$, and such that:*

(i) *If, for all $i < x$, D_i is a 01-shmqd and is a regular bilator, the the product $\prod_{i<x} D_i$ is a regular bilator, and*

$$\prod_{i<x} \mathrm{LIN}^{01}(D_i) = \mathrm{LIN}^{01}\Big(\prod_{i<x} D_i\Big)$$

(ii) *Similarly,*

$$\prod_{i<E_x} \mathrm{LIN}^{01}(\nu^{D_i}_{(f_k)}) = \mathrm{LIN}^{01}(\nu^{\prod_{i<x} D_i}_{(f_k)})$$

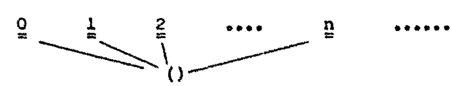
Proof. Add a new colour w, and let $P = \prod^{*w}_{i<w} D_i$; then, in P, remove colour w: replace everywhere $\underset{w}{i}$ by i: the resulting mqd. is by definition $\prod_{i<x} D_i$.

Theorem 7.5.8. *Assume that D is a 0-shmqd of type $(x_k)_{k \in K}$, with $1 \notin K$; then one defines a 01-shmqd $\bigwedge D$ of type $(x_k)_{k \in K}$, which is a regular bilator, in such a way that:*

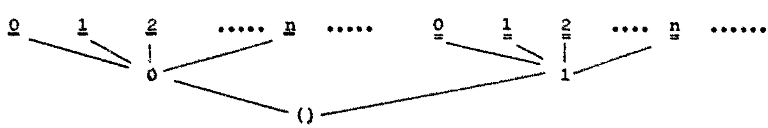
(i) $\mathrm{LIN}^{01}(\bigwedge D) = \bigwedge(\mathrm{LIN}^{0}(D))$.
(ii) $\mathrm{LIN}^{01}(\nu^{\bigwedge D}_{(f_k)}) = \bigwedge(\mathrm{LIN}^{0}(\nu^{D}_{(f_k)}))$.

Proof. the construction is made by induction on $\mathrm{LIN}^{0}(D)$:

(1) if $D = 0$, then $\bigwedge D$ is



(2) if $D = 1 = \{(0)\}$, then $\bigwedge D$ is



(3) if $D = \sum_{z<x} D_z$, then $\bigwedge D = \prod_{z<x} \bigwedge D_z$.
(4) if $D = \sum^{*w}_{i<x_w} D_i$, with $w \neq 0$, then $\bigwedge D = \prod^{*w}_{i<x_w} \bigwedge D_i$,
(5) if $D = \sum^{*0}_{n<\omega} D_n$, then write $\mathrm{SEP}(D) = \sum^{*1}_{n<\omega} D'_n$; add a new colour w, and

replace in D'_n colour 1 by colour w: the result is D''_n. Form

$$P = \prod_{n<\omega}^{*w} (1 + \bigwedge D''_n),$$

then P is easily seen to be $01w$-homogeneous; $\bigwedge D$ is defined to be what remains in P after removing all sequences containing colour 1 (i.e., forming $P^0(\omega, \omega, 0)$) and changing then colour w into colour 1.

The proof of the properties is, as in the previous theorems left to the reader. I don't see any difficulty, except, perhaps, in questions of homogeneity, connected with the verification of (**): but Section 7.6 will give us a simple criterion in that case.

7.6. Finitistic shmqd

Definition 7.6.1. (i) A mqd. D is *finitary* iff all its white branchings are finite: $s \in D^* \to \{x \in \mathrm{ON}, s*(x) \in D^*\}$ is finite.

(ii) A K-shmqd D is *finitistic* iff for all $(x_k)_{k\in K}$, $D((x_k))$ is finitary.

Remark 7.6.2. Call a 0-shmqd strongly finite, weakly finite, according to the associated dilator $\mathrm{LIN}^0(D)$, then

$$D \text{ strongly finite} \to D \text{ finitistic} \to D \text{ finitary} \to D \text{ weakly finite},$$

and all the implications are strict.

Theorem 7.6.3. (i) *If D is finitistic and $D \simeq D'$, so is D'.*

(ii) *If D is 0-shmqd, then D is finitistic iff* $\mathrm{SEP}(D)$ *is finitistic (if $1 \notin K$).*

(iii) *If $D_0, \ldots, D_{n-1}$ are 01-shmqd which are regular bilators, then $\prod_{i<n} D_i$ is finitistic, provided the D_i's are finitistic.*

(iv) *If the 01-shmqd D_i ($i < x_w$) are regular bilators, if the D_i's are finitistic, and if $w \neq 0, 1$, then $\prod_{i<x_w}^{*w} D_i$ is finitistic.*

(v) *If the 0-shmqd D is finitistic, then $\bigwedge D$ is finitistic.*

Proof. Left to the reader; for instance (v) is easily proved by induction on $\mathrm{LIN}^0(D)$, using (ii)–(iv). Finitism of D is used to remark that when $D = \sum_{i<x} D_i$, then x can be supposed to be finite.

Theorem 7.6.4. *If D is a finitistic sh. dendroid, then $\bigwedge D$ is finitistic, and*

(i) $\bigwedge_D(n, m) = \delta_D(n, m)$,

(ii) $\bigwedge_D(m) = \gamma_D(m)$.

Proof. This is essentially Theorems 7.3.3 plus 7.6.3(v); of course a direct proof could be given, using only the direct definition of $\bigwedge D$.

Proposition 7.6.5. *For this proposition, let us speak of homogeneity*, strong homogeneity*, to mean the obvious analogues of these notions when one forgets property (**): then if D is a K-shmqd*, and D is finitistic, then D is a K-shmqd.*

Proof. We give only the main lines: assume that, for some sequence $(x_k)_{k\in K}$, $D^0((x_k))$ is not homogeneous, i.e., does not enjoy (**). From this one gets easily a sequence $(x'_k)_{k\in K}$, together with $s\in D^0((x'_k))^*$, and $(f_k)\in I(x'_k, x'_k)$ such that ($D^0((x'_k))$ is supposed to be normal):

(i) $s\neq m_{(f_k)}(s)$ (hence $s<m_{(f_k)}(s)$),

(ii) if $\underset{k}{z}\in \mathrm{Occ}(s)$, then $f_k(z_k)=z_k$.

Then it is immediate that the sequences $s_n=(m_{(f_k)})^n(s)$ are pairwise distinct; but $\mathrm{Occ}(s^n)=\mathrm{Occ}(s)$.

This is not possible when $D^0((x'_k))$ is finitary, because there can only be finitely many sequences with a given $\mathrm{Occ}(\cdot)$.

Remarks 7.6.6. (i) One can use Proposition 7.6.5 (together with a direct limit argument) to show that the constructions of Section 7.4 do preserve (**).

(ii) But it is also necessary to remark that the natural idea is that of a finitistic shmqd, which corresponds to our intentions and practice.

Appendix. The hierarchy theorem by means of rungs and ladders

We present here a direct argument, using rungs and ladders. This is a mixture of the proofs given in previous versions; features not connected directly to the theorem are omitted.

A.1. *Ordinal rungs*

Definition A.1.1. Let r and a be ordinals, a *rung* R of height r and type a is a 4-tuple $(a, r, T, \cdot[\cdot])$ such that:

(RG 1) T is a function from $r+1$, to $a+1$.

(RG 2) For all $y\leq r$, $y[\cdot]$ is a strictly increasing and continuous function from $T(y)+1$ to $y+1$.

(RG 3) For all y, b such that $y\leq r$ and $b\leq T(y)$:

(i) $T(y[b])=b$,

(ii) if $c\leq b$, then $(y[b])[c]=y[c]$.

(RG 4) If $y\leq r$

(i) $y[T(y)]=y$,

(ii) if y is limit, then $T(y)$ is limit.

(RG 5) Assume that $y, z \leqslant r$, $b < T(y)$, and that $y[b] < z < y[b+1]$, then $y[b] < z[0]$.

Definition A.1.2. The function $y[\cdot]$ from $T(y)+1$ to $y+1$ is the *fundamental sequence* of y (in R).

Definition A.1.3. (i) $\|R\|$ will denote the height of R, whereas $\underline{t}(R)$ will be its type.

(ii) We shall try to follow the following notational principle: we denote a rung by capital letters, and its height by the corresponding small letter: for instance R''_n and r''_n.

Remark A.1.4. Rungs generalize the 'Bachmann collections' [3]; the main difference is the use of fundamental sequences for arbitrary points, not only limit (of course, non-trivial fundamental sequences).

Examples A.1.5. (i) For all ordinals a and b such that $b \leqslant a$, one defines a rung $\underline{b}_a$, of height b and type a, as follows: if $y \leqslant b$, then $T(y) = y$, and if $z \leqslant y$, then $y[z] = z$; $\underline{a}_a$ is abbreviated in $\underline{a}$.

(ii) Suppose that $R = (a, r, T, \cdot[\cdot])$ and $S = (a, s, U, \cdot[\![\cdot]\!])$ are rungs of the same type a, then one defines a rung $R+1+S$ $(=(a, r+1+s, V, \cdot\{\cdot\})$ (the *sum* of R and S) as follows:

(1) if $z < r+1$, then $V(z) = T(z)$; if $c \leqslant T(z)$, then $z\{c\} = z[c]$

(2) if $z \leqslant s$, then $V(r+1+z) = U(z)$; if $c \leqslant U(z)$, then $(r+1+z)\{c\} = r+1+z[\![c]\!]$.

(iii) Suppose that $R = (a, r, T, \cdot[\cdot])$ is a rung, then one defines for $s \leqslant r$ a rung $R \upharpoonright s = (a, s, U, \cdot[\![\cdot]\!])$ simply by restricting T and $\cdot[\cdot]$ to $s+1$.

(iv) Suppose that $R = (a, r, T, \cdot[\cdot])$ and $S = (a, s, U, \cdot[\![\cdot]\!])$ are rungs of the same type a, then we define a rung $R \times S = (a, r \times s, V, \cdot\{\cdot\})$ (the *product* of R and S; $r \times s$ is close to the ordinal product $r \cdot s$):

(1) if $s = 0$, then $R \times S = \underline{0}_a$ (i.e., $R \times \underline{0}_a = \underline{0}_a$),

(2) if $s \neq 0$, then $r \times s = \sup_{z<s}(r \times z + 1 + r + 1)$,
$V(r \times s) = U(s)$, and, if $c \leqslant U(s)$, $(r \times s)\{c\} = r \times (s[\![c]\!])$.

If $z < s$, let $Z = S \upharpoonright z$; then $(R \times S) \upharpoonright (r \times z + 1 + r) = R \times Z + 1 + R$.

(v) Suppose that $R = (a, r, T, \cdot[\cdot])$ and $S = (a, s, U, \cdot[\![\cdot]\!])$ are rungs of the same type a, then we define a rung $(1+R)^S = (a, r^{\cdot s}, V, \cdot\{\cdot\}$ *exponentiation* of rungs; $r^{\cdot s}$ is close to the ordinal exponential $(1+r)^s$):

(1) if $s = 0$, then $(1+R)^S = \underline{1}_a$ (i.e., $(1+R)^{\underline{0}_a} = \underline{1}_a$),

(2) if $s \neq 0$, then $r^{\cdot s} = \sup_{z<s}(r^{\cdot z} + 1 + r^{\cdot z} \times r + 1)$, $V(r^{\cdot s}) = U(s)$, and for $c \leqslant U(s)$, $(r^{\cdot s})\{c\} = r^{\cdot s[\![c]\!]}$; if $z < s$, let $Z = S \upharpoonright z$. Then $(R^{\cdot s}) \upharpoonright (r^{\cdot z} + 1 + r^{\cdot z} \times r) = R^{\cdot Z} + 1 + R^{\cdot Z} \times R$.

Proposition A.1.6. (i) *If* y *and* z *are such that* $y[b]<z[c]<y[b+1]$, *then* $z<y[b+1]$.

(ii) *If* $y[b]=z[c]$ *and* $y\leqslant z$, *then* $b=c$, *and* $y=z[T(y)]$.

Proof. (i) $z=y[b+1]$ is impossible by (RG 3)(ii); suppose that $z>y[b+1]$, then $y[b+1]$ is not of the form $z[d]$ (by (RG 3)(ii) again), so, we have, for some d: $z[d]<y[b+1]<z[d+1]$, with $c\leqslant d$; by (RG 5), one gets $z[d]<y[0]$, so $y[b]<z[c]\leqslant z[d]<y[0]$, contradiction.

(ii) (RG 3)(i) yields $b=T(y[b])=T(z[c])=c$, so $b=c$; if $y=z[d]$ for some d, then $T(y)=d$, $y=z[T(y)]$; otherwise, for some d $z[d]<y<z[d+1]$, and obviously $c\leqslant d$. So $y[0]<y[b]=z[c]\leqslant z[d]<y[0]$ a contradiction.

Proposition A.1.7. *For all* z *such that* $z\leqslant r$, *there is a greatest* $y\leqslant r$ *such that* $z=y[T(z)]$.

Proof. (i) Assume the contrary. Let $y=\sup\{t;\, t[T(z)]=z\}$; by hypothesis, $y[T(z)]\neq z$. As a first consequence, y is limit (since it is the supremum of a family, and does not belong to it). so there is some $b<T(y)$ such that $z<y[b]$, and $t>y[b]$ such that $t[T(z)]=z$; if $t=y[c]$ for some c, then $y[T(c)]=(y[c])[T(z)]=z$; if $y[c]<t<y[c+1]$ for some c with $b\leqslant c<T(y)$, then $z<y[c]<t[0]$, a contradiction.

Proposition A.1.8. (i) *Let* $I_i=[y_i[b_i], y_i[b_i+1][$ $(i=0,1)$ *be such that* $I_0\cap I_1\neq\emptyset$; *then either* $I_0\subset I_1$, *or* $I_1\subset I_0$.

(ii) *Suppose that the interval* I *of* r *is the union of a non void family* $I_i=[y_i[b_i], y_i[$, *then there exists* $y\leqslant r$ *and* $b\leqslant T(y)$ *such that* $I=[y[b], y[$; *furthermore, the interval* $[b, T(y)[$ *is included in the union of the intervals* $[b_i, T(y_i)[$.

Proof. (i) Assume for instance that $y_0[b_0]<y_1[b_1]<y_0[b_0+1]$, then by (RG 5) and Proposition A.1.6(ii) we get $y_0[b_0]<y_1[0]<y_1<y_0[b_0+1]$: so $I_0\supset I_1$, and the extremities of I_1 are distinct from the extremities of I_0.

(ii) Nothing is changed if one assumes that for all i, $T(y_i)=b_{i+1}$. Then, let us treat a particular case: when the intervals I_i are pairwise comparable for inclusion. In that case, observe that (as a consequence of the fact that the extremities are distinct), if $I_i\supset I_j$, then $y_i[b_i]<y_j[b_j]<y_j<y_i$. If the family I_i would contain infinitely many distinct intervals, then one of the sequences $y_i[b_i]$ or y_i would contain a strictly decreasing subsequence. Hence the family contains only finitely many distinct intervals, and the property is proved in that case. In general, it will suffice to consider the case where the intervals I_i are pairwise uncomparable for inclusion. By (i) above, they are necessarily pairwise disjoint. Choose $i\in A$, and let x be maximum such that $x[T(y_i)]=y_i$. Let B be a subset of A (the set of indices) maximal w.r.t the properties:

(1) if $i\in B$, then $y_i=x[T(y_i)]$,

(2) $J=\bigcup_{i\in B} I_i$ is an interval.

Since J is obviously of the form $[x[b], x[c][$, it will suffice to prove that $I = J$, i.e., that $A = B$. Suppose, for instance, that the upper bound $x[c] = y_j[b_j]$ for some j, then by considering $J \cup \{j\}$, one gets a contradiction with maximality of J.

A.2. *Mutilations*

Definition A.2.1. Let $S = (b, s, U, \cdot [\![\cdot]\!])$ be a rung; if $f \in I(a, b)$, then one defines:
(i) A subset S_f^* of S:

$$z \notin S_f^* \leftrightarrow \exists\, x \leq b\, \exists\, u < U(x)(u \notin \mathrm{rg}(f) \text{ and } x[\![u]\!] \leq z < x[\![u+1]\!])$$

(hence $\mathbf{C}S_f^*$ appears as an union of intervals of the form $[y_i[\![b_i]\!], y_i[\![b_i+1]\!][$.)
(ii) An ordinal r = order type of S_f^*, and a function m_f^S (shortly m_f), $m_f^S \in I(r, s)$, defined by $\mathrm{rg}(m_f^S) = S_f^*$. By abuse of notations, we shall write $m_f^S(r) = s$.
(iii) A 4-tuple $f^{-1}(S) = (a, r, T, \cdot [\cdot])$:
(1) $\hat{f}(T(z)) = U(\hat{m}_f(z))$, $(z \leq r)$,
(2) $m_f(z[c]) = (m_f(z))[\![f(c)]\!]$, $(c < T(z), z \leq r)$,
(3) $z[T(z)] = z$, $(z \leq r)$.

Lemma A.2.2. $\hat{m}_f(z[c]) = (m_f(z))[\![\hat{f}(c)]\!]$.

Proof. Let $Z = \hat{m}_f(z[c])$, $Z' = m_f(z[c])$, then $[Z, Z'[$ is a maximal interval in $\mathbf{C}S_f^*$, so, by Proposition A.1.8(ii), this interval is equal to $[Z'[\![u]\!], Z'[$, and $[u, U(Z')[$ is included in $\mathbf{C}\mathrm{rg}(f)$. By maximality, it follows that $u \in \mathrm{rg}(\hat{f})$, and this can only be if $u = \hat{f}(c)$; so

$$\hat{m}_f(z[c]) = Z = Z'[\![\hat{f}(c)]\!] = m_f(z[c])[\![\hat{f}(c)]\!] = m_f(z)[\![\hat{f}(c)]\!]$$

(the last equality is proved by considering separately the trivial case $z = z[c]$, and the case $c < T(z)$. In that case, one uses Definition A.2.1(iii)(2) above).

Theorem A.2.3. (i) *the 4-tuple* $(a, r, T, \cdot [\cdot])$ *of Definition* A.2.1(iii) *is a rung*.
(ii) $f^{-1}(g^{-1}(S)) = (gf)^{-1}(S)$.

Proof. (i) (RG 1), (RG 2), (RG 3)(ii) and (RG 4)(i) are obvious. We verify the other properties:
–(RG 3)(i): by Lemma A.2.2, $\hat{m}_f(z[c]) = m_f(z)[\![\hat{f}(c)]\!]$, so we get:

$$\hat{f}(T(z[c])) = U(\hat{m}_f(z[c])) = U(m_f(z)[\![\hat{f}(c)]\!] = \hat{f}(c),$$

hence $T(z[c]) = c$.
–(RG 4)(ii): if $z \leq r$ is limit, then $\hat{m}_f(z)$ is limit, so we get:

$$\hat{m}_f(z) = \sup_{c < U(m_f(z))} (\hat{m}_f(z)[\![c]\!]);$$

since $U(\hat{m}_f(z)) = f(T(z))$, one gets

$$\hat{m}_f(z) = \sup_{d<T(z)} (\hat{m}_f(z)[\![f(d)]\!]) = \sup_{d<T(z)} (m_f(z)[\![f(d)]\!]) = \sup_{d<T(z)} (m_f(z[d])).$$

So $z = \sup_{d<T(z)} z[d]$.
–(RG 5): if $z[c] < t < z[c+1]$, then $m_f(z[c]) < m_f(t) < \hat{m}_f(z[c+1])$; let $v = m_f(z)$, then

$$v[\![f(c)]\!] < m_f(t) < v[\![f(c)+1]\!].$$

Hence

$$m_f(z[c]_1) = v[\![f(c)]\!] < m_f(t)[\![f(0)]\!] = m_f(t[0]).$$

So $z[c] < t[0]$.
(ii) This is left to the reader.

Remark A.2.4. The proof of Theorem A.2.3 (and Lemma A.2.2) is slightly incorrect, because we have not verified the possibility of defining $f^{-1}(S)$ as in Definition A.2.1(iii); essentially this amounts to show the possibility of Definition A.2.1(iii)(2): this definition is based on the fact that, if $Z \in S_f^*$, then $Z[\![c]\!] \in S_f^*$, for $c \in U(Z)$ iff $c \in \mathrm{rg}(f)$ (immediate).

Proposition A.2.5. *the mutilation 'commutes' to sum, product and exponentiation of rungs.*

Proof. We have to show that $f^{-1}(R+1+S) = f^{-1}(R)+1+f^{-1}(S)$, $f^{-1}(R\times S) = f^{-1}(R)\times f^{-1}(S)$, and $f^{-1}((1+R)^S) = (1+f^{-1}(R))^{f^{-1}(S)}$. This is simply immediate.

Remark A.2.6. The behaviour of mutilation w.r.t. the rungs $\underline{b}_a$ is more complicated: if $f \in I(a', a)$, then $f^{-1}(\underline{a}) = \underline{a}'$; but, $f^{-1}(\underline{b}_a) = \underline{b}'_{a'}$, where b' is defined by $\forall z\ (z < b' \leftrightarrow f(z) < b)$, i.e., $\hat{f}(b') \leq b \leq f(b'$

A.3. *Ladders*

Definition A.3.1. Let K be a finite set; we define the category K-ON as follows:
objects: pairs (x, d), where $x \in \mathrm{ON}$, and d is a function from K to x;
morphisms: the set $I(x, d; y, e)$ of those $f \in I(x, y)$ such that $ef = d$.

Definition A.3.2. Let V be a regular cardinal (of course, everything can be relativized to admissible ordinals) and assume that (V, d) is an object of K-ON. Then one defines:
(i) the category K-ON$\leq(V, d)$:
objects: pairs (x, e) such that $I(x, e; V, d) \neq \emptyset$;
morphisms: $I(x, e; x', e')$;
(ii) the category K-ON$<(V, d)$:
objects: The objects (x, e) of K-ON$\leq(V, d)$, with $x < V$.
morphisms: $I(x, e; x', e')$.

Definition A.3.3. the following data define a category RG:

objects: rungs $(a, r, T, \cdot[\cdot])$;

morphisms: the set $I(R, S)$ consisting of those $f \in I(t(R), t(S))$ such that $f^{-1}(S) = R$.

Definition A.3.4. (i) A *ladder* is a functor from ON to RG, such that:

(1) $t(L(x)) = x$,

(2) $L(f) = f$.

(ii) One defines K-ladders, $K \leq (V, d)$-ladders, $K < (V, d)$-ladders, by replacing ON by K-ON, $K\text{-ON} \leq (V, d)$, $K\text{-ON} < (V, d)$.

Examples A.3.5. (i) $L(x) = \underset{\sim}{x}$ defines a ladder.

(ii) If $k \in K$, then $L(x, d) = \underline{d(k)}_x$ defines a K-ladder.

(iii) Everything obtained by means of sum, product, exponential and Example (i) (resp. (ii)) is a ladder (resp. a K-ladder). For instance, $L(x) = \underset{\sim}{x} + 1 \to \underset{\sim}{x}$ is a ladder, and $L(x, d) = \underset{\sim}{x} \times \underline{d(k)}_x$ is a K-ladder.

Definition A.3.6. Assume that $S = (a, s, U, \cdot[\![\cdot]\!])$ and $R = (s, r, T, \cdot[\cdot])$ are rungs, (the type of R is equal to the height of S); then one defines a new rung $RS = (a, r, V, \cdot\{\cdot\})$ (the *composition* of R and S) by:

(i) $V(x) = U(T(x))$ for all $x \leq r$,

(ii) $x\{d\} = x[T(x)[\![d]\!]]$ for all $d \leq V(x)$.

The verification that RS is a rung is left to the reader.

Definition A.3.7. Assume that L and L' are ladders, then one defines a new ladder $L \circ (\mathrm{Id} + 1 + L')$, as follows:

$$L''(x) = L(x + 1 + \|L'(x)\|)(\underset{\sim}{x} + 1 + L'(x)).$$

(The composition is also defined on K-ladders and on $K < (V, d)$-ladders; in this case, observe that $\|x + 1 + L'(x)\| < V$ for all $x < V$.)

Theorem A.3.8. *The composition maps ladders (resp. K-ladders, $K < (V, d)$-ladders) into themselves.*

Proof. The most interesting case is that of a K-ladder: the proof depends on the following lemma.

Lemma A.3.9. *Let L be a K-ladder, and let $z \leq \|L(x, d)\|$; furthermore assume that $T(z) < x$, and that there is some $u < T(z)$ such that $[u, T(z)] \cap \mathrm{rg}(d) = \emptyset$, then one can find z', $z < z' \leq L(x)$, such that*

$$z'[u] \leq z < z'[T(z) + 1].$$

Proof. Define (x', d') and $f, g \in I(x', d', x, d)$ by:

$$\mathrm{rg}(f) = x - [u, T(z)[, \qquad \mathrm{rg}(g) = x - [u+1, T(z)+1[,$$

then f and g differ only on the argument u. So, if one defines (x'', d'') and $h \in I(x'', d''; x', d')$ by $\mathrm{rg}(h) = x' - \{u\}$, it follows that $fh = gh$. Observe that $\mathrm{rg}(fh) = \mathrm{rg}(gh) = x - [u, T(z)]$. If the conclusion of the lemma is false, then, obviously $z \in L(x, d)^*_{fh} = L(x, d)^*_{gh}$. Define $f^* = m_f^{L(x,d)}$, $g^* = m_g^{L(x,d)}$, $h^* = m_h^{L(x',d')}$, then $f^*h^* = g^*h^*$, and $z \in \mathrm{rg}(f^*h^*)$. If $z = f^*h^*(z')$, and $z' = h^*(z'')$, then $z = f^*(z') = g^*(z')$. We compute the value $T'(z')$ in $L(x', d')$, by means of Definition A.2.1(iii)(1) applied to f and to g:

$$\hat{f}(T'(z')) = T(\hat{f}^*(z')) \quad \text{implies} \quad T'(z') = u,$$
$$\hat{g}(T'(z')) = T(\hat{g}^*(z')) \quad \text{implies} \quad T'(z') = u+1.$$

We have obtained a contradiction.

Proof of Theorem A.3.8. Let $L'' = L \circ (\mathrm{Id} + 1 + L')$, let $S'' = L''(x, d)$, $S' = \underset{\sim}{x} + 1 + L'(x, d)$, $S = L(y, e)$, with $y = \|S'\|$, $e(k) = d(k)$ for all $k \in K$. Assume that $S'' = (x, s'', T'', \cdot\{\cdot\})$, $S' = (x, y, T', \cdot[\![\cdot]\!])$, $S = (y, s'', T, \cdot[\cdot])$: be definition $S'' = SS'$. We prove that $S''^*_f = S^*_g$, where $g = m_f^{S'}$: S''^*_f is obtained by removing all intervals $[z[z'[\![t]\!]], z[z'[\![t+1]\!]][$, where $t \notin \mathrm{rg}(f)$, while S^*_g is obtained by removing all intervals $[z[u], z[u+1][$, where $u \notin \mathrm{rg}(g)$: the interval $[z[z'[\![t]\!]], z[z'[\![t+1]\!]][$, for $t \notin \mathrm{rg}(f)$ is obviously an union of intervals $[z[u_i], z[u_i+1][$, for $u_i \notin \mathrm{rg}(g)$: this shows the inclusion $S^*_g \subset S''^*_f$. Assume that the reverse inclusion does not hold. This means that some interval $[z[t], z[t+1][$, with $t \notin \mathrm{rg}(g)$, is not included in any interval $[b[b'[\![u]\!]], b[b'[\![u+1]\!]][$, with $u \notin \mathrm{rg}(f)$. One can choose z and t such that:
$-z[t]$ is minimum with this property;
$-z$ cannot be written $z'[T(z)]$ for some $z' > z$ (by Proposition A.1.7). Observe that

(i) $T(z) > x$ (otherwise, take $b = z$, $b' = T(z)$, $u = t$);

(ii) $T(z) < y$ (otherwise, let $b = z$, and b' and u such that $b'[\![u]\!] \leq t < b'[\![u+1]\!]$). Then it is possible to apply Lemma A.3.9: $z'[t] \leq z < z'[T(z)+1]$ for some $z' \leq s''$. So, one can write $z'[t'] < z < z'[t'+1]$ for some $t' \leq T(z)$ (the equality $z = z'[t']$ would entail $z = z'[T(z)]$). The interval $[z'[t'], z'[t'+1][$ is not included in $\mathbf{C}S''^*_f$ (since it contains $[z[t], z[t+1][$), but is included in S^*_g. Observe that

(iii) $[t, T(z)+1[\subset \mathrm{rg}(g)$ (let $b = z$, and choose b' and u with $T'(b') = u+1$, and $b'[\![u]\!] \leq t < b'$. It suffices to show that $T(z) < b'$, but, if $b' \leq T(z)$, then $[z[t], z[t+1][$ would be included in $[b[b'[\![u]\!]], b[b'[\![u+1]\!]][$). Finally, $z'[t'] < z[t]$, and this contradicts the minimality of $z[t]$.

The end of the proof offers no difficulty: assume that $f \in I(x', d'; x, d)$, and let

$$R = (y', r'', U, \cdot[\cdot]_1) = f^{-1}(S),$$
$$R' = (x', y', U', \cdot[\![\cdot]\!]_1) = f^{-1}(S'), \qquad R'' = (x', r'', U'', \cdot\{\cdot\}_1) = f^{-1}(S'').$$

It suffices to show that $R''=RR'$ (since $L(y',e')=R$, with $e'(k)=e(k)$ for $k\in K$, $R'=\underline{x}'+1+L'(x',d')$). We get:

$$\begin{aligned}\hat{f}(U''(z))&=T''(\hat{m}_f^{S''}(z))=T'(T(\hat{m}_f^{S''}(z)))=T'(T\hat{m}_g^{S}(z)))\\&=T'(\hat{g}(U(z)))=T'(\hat{m}_f^{S'}(U(z)))=\hat{f}(U'(U(z))).\end{aligned}\tag{1}$$

Hence $U''(z)=U'(U(z))$.

$$\begin{aligned}\hat{m}_f^{S''}(z\{u\}_1)&=\hat{m}_f^{S''}(z)\{\hat{f}(u)\}=\hat{m}_f^{S''}(z)[T(\hat{m}_f^{S''}(z))\llbracket\hat{f}(u)\rrbracket]\\&=\hat{m}_f^{S''}(z)[T(\hat{m}_g^{S}(z))\llbracket\hat{f}(u)\rrbracket]=\hat{m}_g^{S}(z)[\hat{g}(U(z))\llbracket\hat{f}(u)\rrbracket]\\&=\hat{m}_g^{S}(z)[\hat{m}_f^{S'}(U(z))\llbracket\hat{f}(u)\rrbracket]=\hat{m}_g^{S}(z)[\hat{m}_f^{S'}(U(z)\llbracket u\rrbracket_1)]\\&=\hat{m}_g^{S}(z)[\hat{g}(U(z)\llbracket u\rrbracket_1)]=\hat{m}_g^{S}(z[U(z)\llbracket u\rrbracket_1]_1)\\&=\hat{m}_f^{S''}(z[U(z)\llbracket u\rrbracket_1]_1).\end{aligned}\tag{2}$$

Theorem A.3.10. *It is possible to define a function Λ which maps $K\leqslant(V,d)$-ladders on $K<(V,d)$-ladders, and with the properties that:*

(1) if (W,d') is an object of $K<(V,d)$ (W regular cardinal), and if $L'=$ the restriction of L to K-ON$\leqslant(W,d')$, then $\Lambda L'$ is the restriction of ΛL to K-ON$<(W,d)$;

(2) if $L=L'+1+L''$, (obvious notations), then $\Lambda L=\Lambda L'+1+M$ for some $K<(V,d)$-ladder M.

The precise description of Λ is given during the proof.

Proof. By induction on the ordinal $A=\|L(V,d)\|$.

Case 0: $A=0$, then define $\Lambda L(x,e)=(1+\underline{x})^x$; (1) and (2) are trivial.

Case 1: $A\neq 0$, but $T(A)=0$, then it is easy to see that for all (x,e), one can write $L(x,e)=L'(x,e)+1+\underline{0}_x$, for a certain $K\leqslant(V,d)$-ladder L'. When M is a $K<(V,d)$-ladder, let $\vartheta(M)$ abbreviate $M+1+M\circ(\mathrm{Id}+1+M)$; one defines $(\Lambda L)(x,e)=\vartheta(\Lambda L')(x,e)+1+\underline{0}_x$; (1) and (2) are trivial.

Case 2: $A\neq 0$, and $0<T(A)<V$; let $K'=K\cup\{l\}$, where k is a new element. Given e such that (V,e) is an object of K-ON$\leqslant(V,d)$, and $z<T(\|L(V,e)\|)$ (here, we mean T computed in $L(V,e)$) we define a function e_z from K' into V, which is an extension of e, by $e_z(k)=z$, then we define $K'\leqslant(V,e_z)$-ladders L_{ez}, as follows: First observe (Lemma A.3.9) that $T(A)\leqslant\sup(\mathrm{rg}(d)+1)$, so, by mutilation w.r.t. E_{xV}, one gets $T(\|L(x,c)\|)=T(\|L(V,c)\|)$ (we use systematically the same letter T and the same symbol $\cdot[\cdot]$ for the distinct rungs $L(x,c)\ldots$). Assume that $(x,c')\in K'$-ON$<(V,e_z)$, then one sees easily that $c'=c_{z'}$ for some $z'<T(\|L(x,c)\|)=T(\|L(V,c)\|)$. Let $B=\|L(x,c)\|$, then

$$L_{ez}(x,c')=L(x,c)\restriction(B[z'+1]-1)$$

by definition. Observe that

$$\|L_{ez}(V, e_z)\| = A[z+1] - 1 \leq A - 1 < A,$$

so, by induction hypothesis, ΛL_{ez} has been defined. Hence one puts

$$\|(\Lambda L)(x, e)\| = \sup_{z < T(\|L(V,e)\|)} (\|(\vartheta(\Lambda L_{ez}))(x, e_z)\| + 1),$$

$$(\Lambda L)(x, e) \upharpoonright \|(\vartheta(\Lambda L_{ez}))(x, e_z)\| = (\vartheta(\Lambda L_{ez}))(x, e_z),$$

$$U(\|(\Lambda L)(x, e)\|) = T(\|L(x, e)\|) = T(\|L(V, e)\|),$$

$$\|(\Lambda L)(x, e)\| [t] = \sup_{z<t}(\|(\vartheta(\Lambda L_{ez}))(x, e_z) + 1).$$

The verification that this defines a $K<(V, d)$-ladder, and that (1) and (2) hold is immediate.

Case 3: $A \neq 0$, and $T(A) = V$. Observe now that $T(L(x, c)) = x$. If one defines K', L_{ez} as before, then we put:

$$(\Lambda L)(x, e) = \sup_{z<x}(\|(\vartheta(\Lambda L_{ez}))(x, e_z)\| + 1),$$

$$U(\|(\Lambda L)(x, e)\|) = T(\|L(x, e)\|) = x.$$

The rest of the definition is as above; once again, there is practically nothing to verify.

Remark A.3.11. It is clear that in the definition of Λ, everything depends on the choice of ϑ. Many other choices are possible, for instance, take $\vartheta(L) =$ 'iteration' of L, etc... the only principle to satisfy is that $\vartheta(L) = L + 1 + L'$ for some L' depending on L.

A.4. *The hierarchy theorem*

Definition A.4.1. Let $R = (V, r, T, \cdot [\cdot])$ be a rung of type V, V regular cardinal, then we define the following hierarchies of functions from V to V:

(i) if $r = 0$, then

$$\gamma_R(x) = 0, \qquad \lambda_R(x) = x^{\cdot x}$$

(the function $x^{\cdot x}$ is defined in Example A.1.5(v)).

(ii) if $r \neq 0$, and $T(r)$ is successor, then define R' to be $R \upharpoonright r - 1$; then

$$\gamma_R(x) = \gamma_{R'}(x) + 1, \qquad \lambda_R(x) = (\vartheta(\lambda_R))(x) + 1$$

(ϑ is the function $\vartheta(f)(x) = x + 1 + f(x + 1 + f(x))$).

(iii) if $r \neq 0$, and $T(r)$ is limit $< V$, then define R_z, for $z < T(r)$ to be $R \upharpoonright (r[z])$, then

$$\gamma_R(x) = \sup_{z<T(r)} \gamma_{R_z}(x), \qquad \lambda_R(x) = \sup_{z<T(r)} \lambda_{R_z}(x).$$

(iv) if $r \neq 0$, and $T(r) = V$, then define R_x as above: and let

$$\gamma_R(x) = \gamma_{R_x}(x), \qquad \lambda_R(x) = \lambda_{R_x}(x).$$

Proposition A.4.2. *Let L be a $K \leqslant (V, d)$-ladder, and $x > \sup(\mathrm{rg}(d))$, then*
(i) $\gamma_{L(V,d)}(x) = \|L(x, d)\|$,
(ii) $\lambda_{L(V,d)}(x) = \|(\Lambda L)(x, d)\|$.

Proof. The two properties are obtained similarily, for instance, (i): let $A = \|L(V, d)\|$. We argue by induction on A.

(1) If $A = 0$, then $L(V, d) = \underset{\cdot}{0}_V$, and $L(x, d) = E_{xV}^{-1}(\underset{\cdot}{0}_V) = \underset{\cdot}{0}_x$.

(2) If $A = B + 1$, and $T(A) = 0$, then define a $K \leqslant (V, d)$-ladder L' by: $L = L' + 1 + \underset{\cdot}{0}$. Then by induction hypothesis,

$$\gamma_{L'(V,d)}(x) = \|L'(x, d)\|,$$
$$\gamma_{L(V,d)}(x) = \gamma_{L'(V,d)}(x) + 1 = \|L'(x, d)\| + 1 = \|L(x, d)\|.$$

(3) If $A = B + 1$, and $T(A) = t + 1$; then define (with $K' = K \cup \{k\}$, $k \notin K$) d' from K' to V, d' extending d, by $d'(k) = t$. Since we have remarked (Case 2 of the proof of Theorem A.3.10) that $T(A) \leqslant x$, it follows that (x, d') is an object of K'-ON$<(V, d')$. Applying the induction hypothesis to the $K' \leqslant (V, d')$-ladder L' defined by: $L'(V, d') = L(V, d) \restriction B$, one gets:

$$\gamma_{L(V,d)}(x) = \gamma_{L'(V,d')}(x) + 1 = \|L'(x, d')\| + 1 = \|L(x, d)\|.$$

(4) If A is limit, and $T(A) < V$, then act as in (3) above: when $t < T(A)$, define d_t from K' to V, extending d, by $d_t(k) = t$. Define a $K' \leqslant (V, d_t)$-ladder L_t by

$$L_t(V, d) = L(V, d) \restriction A_t, \quad \text{with } A_t = A[t],$$

then

$$\gamma_{L(V,d)}(x) = \sup_{t < T(A)} \gamma_{L_t(V,d_t)}(x) = \sup_{t < T(A)} \|L_t(x, d)\| = \|L(x, d)\|$$

(we have used the fact that $T(L(x, d)) = T(L(V, d))$).

(5) If A is limit, with $T(A) = V$, then define K' and d_t as above, for $t < x$; but we define now the $K' \leqslant (V, d_t)$-ladders L_t by:

$$L_t(V, d_t) = L(V, d) \restriction B_t, \quad \text{with } B_t = A[t+1] - 1,$$

then

$$\gamma_{L(V,d)}(x) = \sup_{t<x} \gamma_{L_t(x,d_t)}(x) + 1 = \sup_{t<x} \|L_t(x, d_t)\| + 1 = \|L(x, d)\|.$$

Corollary A.4.3. *Let L be a ladder, then $\lambda_{L(V)} = \gamma_{(\Lambda L)(V)}$.*

Proof. First we must explain the meaning of 'ΛL'. If one restricts L to the category $\emptyset$-ON$< W$, then ΛL is defined as $a\emptyset < W$-ladder, and its value is in fact independant from the actual choice of W. The proof is an obvious application of Proposition A.4.2.

$$\gamma_{L(V)}(x) = \|(\Lambda L)(x)\| = \gamma_{(\Lambda L)(V)}(x).$$

Remarks A.4.4. (i) In practice, one is interested mainly in the case $V=\omega$; then, in order to compute $(\Lambda L)(\omega)$, one needs to make an induction on the next cardinal ω_1 (in practice: ω_1^{CK}). The hierarchy λ, of functions from ω_1 to ω_1, and indexed by rungs of type ω_1 is (very close to) the Bachmann hierarchy. If one defines

$$L_0(x)=\underline{x}, \qquad L_{n+1}(\underline{x})=(1+\underline{x})^{L_n(x)},$$

then it is easy to show that any provably recursive function of arithmetic is bounded for large values by some function $\lambda_{L_n(\omega)}$. The hierarchy Theorem 4.2.3 shows that $\lambda_{L_n(\omega)}=\gamma_{(\Lambda L_n)(\omega)}$. The height of the rung $(\Lambda L_n)(\omega)$ is given by Definition 4.2.2(ii): this is the value $\lambda_{L_n(\omega_1)}(\omega)$. Clearly, $\|L_n(\omega)\|\to\varepsilon_0$ when $n\to\omega$, whereas one would show easily that $\|\lambda_{L_n(\omega_1)}(\omega)\|\to H$ (the Howard ordinal) when $n\to\omega$.

The tremendous gap between ε_0 and H illustrates plainly the laziness of the hierarchy.

(ii) If one asks the question of the comparison of $\lambda_{(\Lambda L_n)(\omega)}$ with the γ-hierarchy, then the answer is $\gamma_{(\Lambda\Lambda L_n)(\omega)}$. Observe that

$$\|(\Lambda\Lambda L_n)(\omega)\|=\lambda_{(\Lambda L_n)(\omega_1)}(\omega),$$

and that $\|(L_n)(\omega_1)\|={}_{L_n(\omega_2)}(\omega)$; with an abuse of notations:

$$\|(\Lambda\Lambda L_n)(\omega)\|=\lambda_{\lambda_{L_n(\omega_2)}(\omega_1)}(\omega).$$

So, the height of $(\Lambda\Lambda L_n)(\omega)$, when $n\to\omega$, will go up to the (usual) proof-theoretical ordinal of ID_2. In general, the estimate of the provably recursive functions of ID_n (with $\mathrm{ID}_0=\mathrm{PA}$) by means of the hierarchy γ will need the ordinal which is associated with ID_{n+1}, but in terms of γ.

(iii) The morality is that we have proved that the Bachmann hierarchy (or an inessential variant) is functorial, i.e., that all Bachmann hierarchies (including hierarchies of functions from ω to ω) from any regular cardinal to itself, 'commute with mutilation'. This means that $\lambda_{L(V)}(x)=\lambda_{L(x^+)}(x)$, when L is a ladder.

The operation Λ can be therefore be understood as a rewriting of the Bachmann hierarchy, not as a function mapping Bachmann collections of type V into functions from V to V, but as an inner operation on ladders: to L associate ΛL.

(iv) If one adds to these facts the remark that a ladder commutes to direct limits and to pull-backs, it will follow that, in the definition of the Bachmann hierarchies, only the case $V=(\omega)$ matters (since it will be possible to express $(\Lambda L)(x)$ as $\varinjlim^*((\Lambda L)(n), f_{nm})$, when $x=\varinjlim^*(n, f_{nm})$, and $(\Lambda L)(n)=\lambda_{L(\omega)}(n)$).

(v) Perhaps (iv) is a bit puzzling, because one is tempted to conclude that the higher number classes are of no use for the Bachmann hierarchy. the answer is more subtle:

– the higher number classes are not needed to *define* the values $\lambda_{L(V)}(x)$, since this can be done by direct limits,

–but, if one wants to show that the values $\lambda_{L(V)}(x)$ are *well founded*, then one needs the well-foundedness of $L(x^+)$.

(vi) One last word: if one forgets 'the extra structure', a ladder induces a dilator. There is no simple characterization of those dilators which are obtained from ladders, and it is not even true that two distinct ladders induce distinct dilators. Furthermore, one of the most interesting technique of dilators, i.e., separation of variables cannot be performed on ladders, i.e., if D comes from a ladder, then $\mathrm{SEP}(D)^n$ will not in general correspond to a ladder.[1] The concept corresponding to dilators is dendroids, and is not as simple as the concept of ladder.

At that time I have reasonable hopes of finding soon a concept which has the advantages of both approaches without the inconvenients.

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